Mapping Problems in the Calculus of Variations: Twists, L^1-Local Minimisers and Vectorial Symmetrisation

by

Charles Graham Morris

Submitted for the degree of

Doctor of Philosophy

in the

Department of Mathematics

University of Sussex

April 2017

Declaration

I hereby declare that this thesis has not been, and will not be, submitted in whole or in part to another University for any other degree or academic award.

Signature: ....

Charles Morris

University of Sussex

Mapping Problems in the Calculus of Variations: Twists, L^1-Local Minimisers and Vectorial Symmetrisation

Charles Graham Morris

Submitted for the degree of

Doctor of Philosophy

Thesis Summary

This thesis has two main yet deeply interconnected trends. Firstly the utilisation of twist mappings in certain geometric problems in the multi-dimensional Calculus of Variations and secondly the study of the spectral counting function of the Laplace-Beltrami operator for a compact Lie groups with particular emphasis on the orthogonal and unitary groups. The link between the two comes from the intimate relation between the geodesics on the Lie group $\mathbf{SO}(N)$ defining twists maps and spectrum of the Laplacian. In the first problem we primarily focus on energy functional of the type,

$$\mathbb{E}[u] = \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla u|^2 \, dx, \tag{0.0.1}$$

when $0 < c \leq \mathrm{F} \in \mathbf{C}(\mathbb{R}^2)$ and $\mathbf{X} \subset \mathbb{R}^n$ (with $n \geq 2$) is some Lipschitz bounded domain. Here the class of admissible mapping which we consider these energies over is

$$\mathcal{A}(\mathbf{X}) = \Big\{ u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u = 1 \ \ a.e \text{ and } u = x \text{ on } \partial \mathbf{X} \Big\}. \tag{0.0.2}$$

We can think of $\mathcal{A}(\mathbf{X})$ as the class of incompressible deformations of a hyperelastic material occupying $\mathbf{X}$ and (0.0.1) as the energy cost of the deformation. For most parts of this thesis we focus on the case where $\mathbf{X} = \{x : a < |x| < b\}$ is an n-dimensional annulus and seek conditions under which twist mappings ($u(x) = \mathrm{Q}(r)x$ where $\mathrm{Q} \in \mathbf{C}([a, b], \mathbf{SO}(n))$ with $r = |x|$) provide extremals and strong local minimisers of $\mathbb{E}$ over $\mathcal{A}(\mathbf{X})$. For example, when $n = 2$ we prove, by exploiting the rich homotopy structure of $\mathcal{A}(\mathbf{X})$, that twist mappings give rise to countably many strong local minimisers provided that F satisfies some constraints. Moreover, we show that in each homotopy class there is a twist mapping which is the unique minimiser of $\mathbb{E}$. To achieve this we develop symmetrisation techniques for vector-valued mappings which not only decreases the energy of $\mathbb{E}$, but also maintain the homotopy class of the initial mapping.

The second theme of this thesis has deeper roots in geometric analysis of compact Lie groups and connects well with the classical question of "*hearing the shape of a drum*": what geometric information about a Riemannian manifold (M^d, g) can we extract purely from the knowledge of its spectrum? A decisive tool here is the spectral counting function defined by $\mathscr{N}(\lambda; \mathbb{G}) = \sharp\{j \in \mathbb{N} : \lambda_j \leq \lambda\}$ and its asymptotics as $\lambda \nearrow \infty$. A celebrated result of L. Hörmander asserts that for a general compact boundryless manifold (M^d, g) the spectral counting function has the asymptotics

$$\mathscr{N}(\lambda; \mathbb{G}) = \frac{\mathrm{Vol}_g(M^d)\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{d-1}{2}}\right). \qquad \lambda \nearrow \infty, \tag{0.0.3}$$

Motivated by the earlier work in this thesis, combined with the deep connection between periodic geodesics and the sharpness of (0.0.3), we ask if the asymptotics of (0.0.3) for a compact Lie groups is sharp? Through our study we show that the remainder term in (0.0.3) can be improved for general compact Lie groups $\mathbb{G}$ provided that $n = \mathrm{Rank}(\mathbb{G}) \geq 2$. Additionally, when $\mathbb{G}$ is one of the orthogonal or unitary groups the estimate on the remainder can be improved to $O(\lambda^{(d-2)/2})$ provided that $\mathrm{Rank}(\mathbb{G}) \geq 8$. We obtain this sharp result by using number theoretical tools similar to those used for the classical Gauss circle problem in dimensions five and above.

Acknowledgements

Firstly, I would like to thank my supervisor Dr Ali Taheri for his help and unwavering support throughout my studies. His passion and enthusiasm for mathematics has been a constant source of inspiration and motivation for me. Furthermore, his guidance over the past four years has been both needed and immensely helpful and for this I remain truly indebted to him.

The University of Sussex has been my home for the past seven years, during this time I have gained many happy memories that will stay with me throughout my life. The support which I have received from all of the staff and faculty at the university is very much appreciated. I have also been fortunate enough to receive financial funding from ESPRC and the University of Sussex for which I would like to acknowledge and express my gratitude for.

When I started my studies in mathematics as an undergraduate, I had the good fortune to meet Kiresh Parmar whose friendship over the last seven years has been priceless. Further I wish to also mention some of my friends who I have shared an office with over my studies; namely Stuart Day, who has been a constant since the beginning of my PhD and more recently George Morrison, George Simpson and Wakil Sarfaraz. Let me also take this opportunity to thank every member of the Selsey Bubbles for all the great memories we have shared.

After finishing my degree in Mathematics at University of Sussex I had the pleasure of receiving the MacQuitty prize. I would like to take this moment again to express my sincere thanks to Dr Jonathan MacQuitty for his extreme generosity which truly enabled me to freely pursue my academic studies.

I would also like to thank Dr Andrew Morris for kindly extending me the opportunity to speak about some of my work at the University of Birmingham analysis seminar. The warm reception and generous hospitality I received during my trip to Birmingham is greatly appreciated.

Additionally let me express my gratitude again to the organisers of conferences which I have attended throughout my PhD. In particular, the 6th South West Regional PDEs Winter School at the University of Oxford (2014), Harmonic Analysis conference at ICMS and the University of Edinburgh (2015) and the winter school of Nonlinear Function Spaces in Mathematics and Physical Sciences held at the University of Lyon 1 (2015). Furthermore I wish to mention and show my appreciation to Prof Jey Sivaloganathan, Prof Jan Kristensen, Dr Arghir Zarenscu and Prof Jonathan Bennett for beneficial and helpful discussions.

As with anything I have achieved in life a special thank you must go to my loving parents. The support and encouragement I have received from you both is, as always unconditional and something which I am eternally grateful for. In addition, I would also like to mention and thank my three sisters Elaine, Jacqui and Becky for their love and care over the years. I must also thank my in-laws for not only welcoming me into their family but for assisting me through my studies.

Finally, my most important thank you goes to my beautiful wife Minnie for everything she does for me. Through all of the tough times she is there to pick me up with a loving smile and a caring heart. I certainly feel blessed to have you by my side and know that I would not have been able to complete this thesis without you.

List of publications with author contribution

- **Twist maps as energy minimisers in homotopy classes: symmetrisation and the coarea formula**
  Authors: Charles Morris and Dr Ali Taheri.
  Journal of Nonlinear Analysis, Vol. 152, 2017, pp. 250-275.
  - Charles Morris: Completed and conceived the objectives and overall goals of the paper. Developed the *novel* techniques used to prove the minimality of twist mappings. Supplied the vast majority of mathematical calculations and arguments needed in the proofs of new results. Created the first draft of the paper and subsequently worked on all of the revised versions.
  - Dr Ali Taheri: Set the plans and goals for the paper. Contributed heavily to the section on existence of L^1-local minimiser and the overall writing/presentation of the work. Also closely supervised the efforts of Charles Morris.
- **Whirl Mappings on Generalised Annuli and Incompressible Symmetric Equilibria of the Dirichlet Energy**
  Authors: Charles Morris and Dr Ali Taheri.
  Submitted and under review 2017.
  - Charles Morris: Contributed to the overall goals and task for the paper. Provided the majority of calculations and mathematical arguments for most proofs of stated result. Developed the analysis of *whirls* in higher dimensions. Created the first draft of the paper and subsequently worked on all of the revised versions.
  - Dr Ali Taheri: Conceived the major goals for the paper and contributed to the writing/revising of the overall work. Closely supervised the work of Charles Morris whilst also working on the analysis. .
- **On the Uniqueness of Energy Minimisers in Homotopy Classes**
  Authors: Charles Morris and Dr Ali Taheri.

To be submitted for publication 2017.

- Charles Morris: Planned the structure and conceived most of the goals and aims of the paper. *Improved* on previous techniques of paper one and further developed *symmetrisation* arguments needed for the proof of twist minimality. Provided most of calculations and mathematical arguments where for most proofs of claimed result. Created the first draft of the paper and subsequently worked on all of the revised versions.
- Dr Ali Taheri: Contributed to the planned goals for the paper and to the writing/revising. Closely supervised the work of Charles Morris and help with the mathematical analysis and arguments.

• **Incompressible twists as limits of compressible local energy minimisers on annuli: A Γ-convergence approach**
Authors: Charles Morris and Dr Ali Taheri.
To be submitted for publication 2017.

- Charles Morris: Helped plan the goals and structure of the paper. Developed the *symmetrisation* arguments from previous papers to the compressible model setting whilst connecting to alternative methods used by other authors. *Improved* on previous existence results of twist critical points for polyconvex energies. Completed most of the calculations and mathematical arguments needed throughout the paper. Created the first draft of the paper and subsequently worked on all of the revised versions.
- Dr Ali Taheri: Conceived the major goals for the paper and contributed to the writing/revising of the overall work. Helped heavily with the mathematical arguments needed to proof existence of twist solutions. Closely supervised the work of Charles Morris.

• **On Weyl's asymptotics and remainder term for the orthogonal and unitary groups**
Authors: Charles Morris and Dr Ali Taheri.
Journal of Fourier Analysis and its Applications, Vol. 23, No. 3, 2017, pp. 1-28.

- Charles Morris: Helped conceive the goals and structure of the paper. Implemented the arguments needed to show the *improvement* in the asymptotics of the spectral counting function for the orthogonal and unitary groups. Developed the approach needed to gain the *sharp* asymptotic result for these groups. Completed most of calculations and mathematical arguments needed throughout the paper. Helped created the first draft of the paper and subsequently worked on all of the revised versions.

– Dr Ali Taheri: Conceived the major goals for the paper and contributed to the writing/revising of the overall work. Helped heavily with the overall presentation and structure of the paper. Closely supervised the work of Charles Morris.

To my wife and parents.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Background and Motivation

The roots of this thesis lie in the study of geometrical problems from the multidimensional Calculus of Variation. It is not uncommon throughout science to encounter the problem of trying to minimise some energy cost functional over an appropriate space. For instance in the field of solid mechanics, and in particular nonlinear elasticity theory, one often encounters a bulk energy of the following type,

$$\mathscr{E}[u;\mathbf{X}] = \int_{\mathbf{X}} W(x,\nabla u)\,dx, \tag{1.1.1}$$

which represents the cost of deforming a bounded elastic body $\mathbf{X} \subset \mathbb{R}^n$ by the deformation $u : \mathbf{X} \to \mathbb{R}^n$. To avoid interpenetration of matter the stored energy density, given by, $W : \mathbf{X} \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ is assumed to satisfy $W \equiv \infty$ for $\det \mathbf{F} \leq 0$. This means that it is only

$$\mathbb{R}^{N\times N}_{+} = \{\mathbf{F} \in \mathbb{R}^{N\times N} : \det \mathbf{F} > 0\}$$

of interest and therefore one suitable class of competing deformations which can be considered are,

$$\mathcal{A}^{+}(\mathbf{X}) = \{u \in W^{1,p}(\mathbf{X},\mathbb{R}^n) : \det \nabla u > 0 \ a.e. \text{ and } u \equiv \varphi \text{ on } \partial\mathbf{X}\}, \tag{1.1.2}$$

where φ denotes some relevant boundary condition placed on the deformations, with equality meant in the sense of traces. Furthermore the stored energy function $W : \mathbf{X} \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ which one can often encounter are polyconvex meaning that $W(x,\mathbf{F})$ is a convex function of all minors of $\mathbf{F}$. To be more precise we mean that there exists a function $\Phi : \mathbf{X} \times \mathbb{R}^{\tau(n)} \to \mathbb{R} \cup \{\infty\}$ such that for almost every $x \in \mathbf{X}$ and every $X \in \mathbb{R}^{\tau(n)}$ the map,

$$X \to \Phi(x,X), \tag{1.1.3}$$

is convex and moreover $W(x, \mathbf{F}) = \Phi(x, T(\mathbf{F}))$ where,

$$T(\mathbf{F}) = (\mathbf{F}, \mathrm{adj}_2\mathbf{F}, \cdots, \mathrm{adj}_N\mathbf{F}). \tag{1.1.4}$$

Note that $\mathrm{adj}_s\mathrm{F}$ stands for the matrix of all $s \times s$ minors of $\mathbf{F}$ and $\tau(n) = \sum_{j=1}^{n} \binom{n}{s}^2$. The assumption that the stored-energy function W is polyconvexity in this setting is often more natural physically than the stronger assumption of convexity of W (for more on this the reader is refereed to [6] and [7]). In addition to this polyconvexity of W it can often occur that the stored energy function W is isotropic and frame indifferent in the sense that,

$$W(x, \mathrm{Q}\mathbf{F}) = W(x, \mathbf{F}\mathrm{Q}) = W(x, \mathbf{F}), \tag{1.1.5}$$

for all $\mathrm{Q} \in \mathbf{SO}(n)$ and $\mathbf{F} \in \mathbf{R}^{n\times n}$ (for further discussion and representations of isotropic and frame indifferent stored energy functions W *cf.*, e.g., [1, 6, 10, 23] or [24]). A prototypical example of such a stored energy functional W is given by,

$$W(x, \mathbf{F}) = W(\mathbf{F}) = tr\{\mathbf{F}^t\mathbf{F}\} + h(\det \mathbf{F}) = \sum_{j=1}^{N} v_j^2 + h(\prod_{j=1}^{N} v_j), \tag{1.1.6}$$

where h is any convex function on the line, $v_1, ..., v_N$ are the singular values of $\mathbf{F}$, that is, the eigenvalues of $\sqrt{\mathbf{F}^t\mathbf{F}}$. Note also the second identity assumes that $\det \mathbf{F} > 0$ and furthermore the representation of W by the singular values is due to the invariance under $\mathbf{SO}(n) \times \mathbf{SO}(n)$, i.e. being isotropic and frame invariant (*see*, [1, 7, 9, 10, 11] and [23, 24] for more). The existence of minimisers to $\mathscr{E}$ over the class of admissible mappings $\mathcal{A}^+(\mathbf{X})$ when $p > n$ is known provided that $\exists u \in \mathcal{A}^+(\mathbf{X})$ such that $\mathscr{E}[u; \mathbf{X}] < \infty$ and Φ suitably nice, i.e. a Caratheodory function, whilst also satisfying the coercivity condition,

$$W(x, \mathbf{F}) \geq f(x) + c|\mathbf{F}|^p, \tag{1.1.7}$$

for some $f \in L^1(\mathbf{X})$ and $c > 0$. This existence result is due to the seminal work of J. Ball in [6] and further results of this type can also be found in [24]. Once the existence of minimisers to this type of problem is achieved one can also ask the question of the existence and multiplicity of local minimisers to the stored energy functional $\mathscr{E}$ over $\mathcal{A}^+(\mathbf{X})$. The term local minimiser has different meanings depending on the context but in this thesis we are explicitly concerned with L^1-local minimisers. A map $u \in \mathcal{A}^+(\mathbf{X})$ is referred to as an L^1-local minimiser *iff* there exists $\delta[u] > 0$ such that for all $v \in \mathcal{A}^+(\mathbf{X})$ with $\|u - v\|_1 \leq \delta$ implies that,

$$\mathscr{E}[u; \mathbf{X}] \leq \mathscr{E}[v; \mathbf{X}]. \tag{1.1.8}$$

One approach to try and prove the existence of multiple local minimisers is to try and exploit, if possible, some of the topological structure of the underlying space $\mathcal{A}^+(\mathbf{X})$ as in [63, 74, 75] and [76]. To be more concrete let us focus on the situation which is familiar to the ones we consider in this thesis. Namely suppose that the domain $\mathbf{X}$ is a 2-dimensional annulus, i.e. $\mathbf{X} = \mathbf{X}[a, b] = \{x \in \mathbb{R}^2 : a < |x| < b\}$ for some $0 < a < b < \infty$. Moreover let the space of admissible mappings be incompressible self-mapping of $\mathbf{X}$, i.e.

$$\mathcal{A}(\mathbf{X}) = \{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^2) : \det \nabla u = 1 \ a.e. \text{ and } u \equiv x \text{ on } \partial\mathbf{X}\}. \tag{1.1.9}$$

It is known from [35] or [36] that any $u \in \mathcal{A}(\mathbf{X})$ has, again denoted by u, a continuous representative $u \in \mathbf{C}(\mathbf{X}, \mathbb{R}^2)$. In addition to this due to the added identity boundary condition constraint the continuous representative u lies in,

$$\mathscr{C}(\mathbf{X}) = \{u \in \mathbf{C}(\mathbf{X}, \mathbb{R}^2) : u(\overline{\mathbf{X}}) = \overline{\mathbf{X}} \text{ and } u \equiv x \text{ on } \partial\mathbf{X}\}. \tag{1.1.10}$$

Therefore each mapping $u \in \mathcal{A}(\mathbf{X})$ has a representative which is a continuous self-mapping of the 2-dimensional annulus. The space $\mathscr{C}(\mathbf{X})$ has a rich homotopy structure which has been also well studied by A. Taheri in [74, 75] and [76]. Exploiting this structure one can write,

$$\mathcal{A}(\mathbf{X}) = \bigcup_{k \in \mathbb{Z}} \mathscr{A}_k(\mathbf{X}), \tag{1.1.11}$$

where $\mathscr{A}_k(\mathbf{X})$ are pairwise disjoint and sequentially weakly closed with respect to $W^{1,2}$. Thus provided that $\exists v \in \mathscr{A}_k$ such that $\mathscr{E}[v; \mathbf{X}] < \infty$ and W satisfies the following coercivity constraint,

$$W(x, \mathbf{F}) \geq f(x) + c_1|\mathbf{F}|^2, \qquad \mathbf{F} \in \mathbb{R}^{2\times 2} : \det \mathbf{F} = 1, \tag{1.1.12}$$

where $f \in L^1(\mathbf{X})$, $c_1 > 0$ then $\exists u_k \in \mathscr{A}_k$ such that,

$$\mathscr{E}[u_k; \mathbf{X}] = \inf_{u \in \mathscr{A}_k} \mathscr{E}[u; \mathbf{X}]. \tag{1.1.13}$$

Furthermore we shall see later in the thesis that being a minimiser of $\mathscr{E}$ in $\mathscr{A}_k$ is enough to obtain that u_k is a L^1-local minimiser of $\mathscr{E}$ in $\mathcal{A}(\mathbf{X})$. Motivated by this existence of local minimiser above, and previous works in [63, 64, 65, 66, 74, 75, 76] for a more general context, we study in this thesis what *features* these local minimisers possess. Namely, given a local minimisers u of an energy of the form of $\mathscr{E}$ what can we say about its *regularity*, global *invertibility*, *symmetry* properties, *etc.* (see [11]). In particular we shall be focused on studying whether local minimiser with certain rotational symmetry properties exist.

1.2 Thesis Overview

Let us now be more precise about the work carried out in this thesis and briefly summarise each chapter with the corresponding contribution made by the work.

- **Chapter 2: Twist maps as energy minimisers in homotopy classes: symmetrisation and the coarea formula**

  This paper is based on work which was carried out at the end of my first year as a PhD student. In it we consider a particular energy of the form

  $$\mathbb{F}[u;\mathbf{X}] = \int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2}\,dx,$$

  where $\mathbf{X} \subset \mathbb{R}^n$ is an n-dimensional annulus and our space of admissible mappings is given by the class of incompressible Sobolev maps

  $$\mathcal{A}(\mathbf{X}) = \left\{ u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u = 1 \ \textit{a.e.} \text{ in } \mathbf{X} \text{ and } u|_{\partial \mathbf{X}} = \phi \right\}.$$

  Here ϕ denotes the identity mapping and $u|_{\partial \mathbf{X}} = \phi$ in the sense of traces on $\partial \mathbf{X}$. In the particular setting that $n = 2$ the energy $\mathbb{F}$ over $\mathcal{A}(\mathbf{X})$ is closely connected to the distortion function from Geometric Function Theory and this relation is explored in detail during section six. The purpose of this paper however, which was motivated by the previous work in [74, 75], is to investigate whether the class of mappings called *twists* provide L^1-local minimisers to $\mathbb{F}$ over $\mathcal{A}(\mathbf{X})$. Through the course of the paper we show that the answer to this question is positive and in fact there are countable many L^1-local minimiser of the form of a twist. This result amounts to a significant improvement on the previous works in [74, 75]. We also prove that in higher dimensions ($n > 2$) twists provide countably many critical points to $\mathbb{F}$ when n is *even* and strikingly however this reduces to only one when n is *odd*. We finish this paper by defining twist mappings in the case of self-mappings of the solid and thickened torus in $\mathbb{R}^3$. In the setting of the solid torus we prove that the only twist mapping $u = \mathrm{Q}[g]x$ which is a classical solution to the Euler-Lagrange associated to $\mathbb{F}$ is the trivial identity mapping. However in the context of the thickened torus we prove that a twist mapping is a critical point to $\mathbb{F}$ provided that the corresponding *angle of rotation* function g satisfies a reduced partial differential equation.

- **Chapter 3: Whirl Mappings on Generalised Annuli and Incompressible Symmetric Equilibria of the Dirichlet Energy**

  The purpose of this paper is to study whether a particular class of mappings which we call *whirls* give rise to any equilibria of the Dirichlet energy. To be more precise we consider a similar geometrical setup to the previous paper where $\mathbf{X}$ denotes an n-dimensional annulus and the Dirichlet energy $\mathbb{E}$ is consider over the class of incompressible self-mappings given by $\mathcal{A}(\mathbf{X})$. Inspired by the work in [75]-[77] we set ourselves the task of finding classical solutions to the Euler-Lagrange system associated to $\mathbb{E}$ over $\mathcal{A}(\mathbf{X})$. To this end we introduce the class of mappings called whirls, which like twists, possess a large amount of symmetry. For instance, when $n = 3$ a whirl mapping u is symmetric about the x_3 axis. Through the process of the paper we show, in a similar spirit to twist mappings, that only the trivial whirl mapping (identity mapping) is a critical point to the Dirichlet energy when $n = 3$. This pattern also continues into the higher dimensional setting where we prove that whirls provide countably many critical points to $\mathbb{E}$ if n is even, whereas in odd dimensions we only obtain the single trivial whirl solution. The work in this chapter makes a modest contribution to the understanding of symmetry properties of critical points to the Dirichlet energy in above stated context.

- **Chapter 4: On the Uniqueness of Energy Minimisers in Homotopy Classes** This paper continues and significantly improves upon the results obtained when $n = 2$ in Chapter 2. Throughout the paper we only consider this planar setting where $\mathbf{X} \subset \mathbb{R}^2$ is an annulus and the class of admissible mappings of interest is again $\mathcal{A}(\mathbf{X})$. The goal of the paper, which is inspired by Chapter 2, is to prove that twist mappings provide countably many L^1-local minimisers to a wide family of energies defined by,

$$\mathscr{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2) |\nabla u|^2 \, \mathrm{d}x. \tag{1.2.1}$$

Here the function $0 < c \leq \mathrm{F} \in \mathbf{C}^\infty(\mathscr{R})$ where $\mathscr{R} = \{x \in \mathbf{X} : x_1, x_2 > 0\}$. To achieve our desired goal we develop a symmetrisation argument which associates to each mapping $u \in \mathcal{A}(\mathbf{X})$ a twist mapping with less energy in $\mathscr{F}$ whilst remaining close to u in a homotopic sense. Namely, it is well known fact that $\mathcal{A}(\mathbf{X}) = \bigcup_{k \in \mathbb{Z}} \mathscr{A}_k$ with $\mathscr{A}_k$ being pairwise disjoint homotopy classes. The symmetrisation technique which is developed in the paper allows us to take any mapping $u \in \mathscr{A}_k$ and define from it a twist mapping $\bar{u} \in \mathscr{A}_k$ where the corresponding energy $\mathscr{F}$ has been decreased. This symmetrisation however does rely on some constraints being satisfied by the function F in order for the energy $\mathscr{F}$ to be decreased. When these constraints are satisfied we use the symmetrisation to show that there exists a twist mapping $u_k \in \mathscr{A}_k$ for each

$k \in \mathbb{Z}$ such that u_k is the unique minimiser of $\mathscr{F}$ over $\mathscr{A}_k$. Furthermore, we prove in the paper that any mapping $u \in \mathcal{A}(\mathbf{X})$ is a Sobolev homeomorphism. A consequence of this is that the class of functions F for which we can prove uniqueness of twist minimisers in each homotopy class $\mathscr{A}_k$ can be significantly increased. The work in this paper goes far beyond what was achieved in Chapter 2 and previous work in [64], [75] and [76].

- **Chapter 5: Incompressible twists as limits of compressible local energy minimisers on annuli: A Γ-convergence approach**

  The work in this paper contains our most recent results and is based in the setting of compressible mappings of an n-dimensional annulus. In particular we consider polyconvex energies of the form,

$$\mathbb{E}_\varepsilon[u] = \int_{\mathbf{X}} \left[\frac{1}{2} \mathrm{F}(|x|^2, |u|^2)|\nabla u|^2 + \frac{1}{\varepsilon} h(\det \nabla u) \right] dx,$$

  where $\varepsilon > 0$, $0 < c \leq \mathrm{F} \in \mathbf{C}^\infty(\mathscr{R})$ and $0 \leq h \in \mathbf{C}^2(\mathbb{R}_+)$ is a convex function which satisfies $h(1) = 0$. As mentioned above we consider this energy over the class of compressible mappings of the annulus $\mathbf{X} \subset \mathbb{R}^n$ given by,

$$\mathcal{A}^+(\mathbf{X}) = \{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u > 0 \ a.e. \text{ and } u \equiv x \text{ on } \partial \mathbf{X}\}.$$

  The purpose of this paper is three fold. Firstly, motivated by previous work in [74], [75], [76], [78] and in particular [66] we wish to study the existence of twist mappings as classical solutions to the Euler-Lagrange equations associated with the energy $\mathbb{E}_\varepsilon$ in the general n-dimensional setting over $\mathcal{A}^+(\mathbf{X})$. In this regard we show, in the early part of the paper, that there are countably many twist solutions provided the dimension of the underlying annulus is even. However, when the dimension of the annulus is odd we are only able to prove the existence of one critical twist mapping. The second purpose of this paper is to consider, when $n = 2$, the incompressible model from Chapter 4 as the limit problem of the above as $\varepsilon \to 0$. We know from Chapter 4 that the limit problem has a unique minimiser in each homotopy class, i.e. $u_k \in \mathscr{A}_k$. Therefore, we enquire if the minimisers in the homotopy classes of the compressible mapping, i.e. $u_\varepsilon^k \in \mathscr{A}_k$, converge in some sense to the limit minimiser u_k. Note that the class of compressible mappings in two dimensions can be decomposed into its homotopy classes like in the incompressible case, i.e. $\mathcal{A}^+(\mathbf{X}) = \bigcup_{k \in \mathbb{Z}} \mathscr{A}_k^+$, and furthermore $\mathscr{A}_k \subset \mathscr{A}_k^+$. The results of our study show that the minimisers converge strongly in $W^{1,2}$ as $\varepsilon \to 0$ and moreover there is a sequence of critical twist mappings which converge strongly in $W^{1,2}$ to u_k. The underlying principle to this analysis is that the incompressible model from Chapter 4 is the Γ-limit of the above compressible model as $\varepsilon \to 0$. The final goal of this paper is to see, again when $n = 2$, if the symmetrisation techniques which were developed in Chapter 4 can be extended to the compressible mapping setting. Namely, are twists minimisers of $\mathbb{E}_\varepsilon$ for $\varepsilon > 0$ in the homotopy classes $\mathscr{A}_k^+$. This final goal again requires further development of the symmetrisation techniques of Chapter 4 and leads us to introduce what we call the *annular rearrangement* of a measurable function $f : \mathbf{X} \to \mathbb{R}$. In doing this we are able to show that provided F satisfies the assumptions of Chapter 4 and the convex function h also satisfies some suitable assumptions we obtain the existence of a twist minimiser in each homotopy class $\mathscr{A}_k^+$ for any $\varepsilon > 0$. In summary, the work in this paper is a natural progression of the previous work in Chapter 4 and a significant advancing of the work done in [66].

- **Chapter 6: On Weyl's asymptotics and remainder term for the orthogonal and unitary groups**

  The final paper to be included in this thesis is quite different in nature to the chapters proceeding it. However, it does firmly have its motivation from the work above and in particular twist mappings. Motivated by the significance of the period geodesics on the Lie group $\mathbb{G} = \mathbf{SO}(N)$ in providing twist solutions to certain geometric problems in the calculus of variations (i.e. Chapters 2-5) we take a closer look at the periodic geodesics on $\mathbb{G} = \mathbf{SO}(N)$. During this study we try to quantify the amount of periodic geodesics in the sense of a geodesic length counting function. Through this and the *deep* connection between the *geodesic flow* on a compact Riemannian manifold (M^d, g) and the corresponding spectral counting function of the manifold we are lead naturally to the study of the latter. The spectral counting function for a compact Riemannian manifold (M^d, g) is defined by,

$$\mathscr{N}(\lambda; M^d) = \sharp\{j \geq 0 : \lambda_j \leq \lambda\}, \qquad \lambda > 0, \tag{1.2.2}$$

  and where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$ denote the eigen-values of $-\Delta_g$ in ascending order. Note that each eigen-value has a finite multiplicity while $\lambda_j \nearrow \infty$ as $j \nearrow \infty$. This spectral counting function has been the focus of many studies and in particular is an important tool when one tries to study the famous question of "*hearing the shape of a drum*". Meaning what geometric information can we glean about (M^d, g) purely from the knowledge of its spectrum? Of particular interest from many notable authors over the years has been the study of the asymptotic behaviour of $\mathscr{N}(\lambda; M^d)$

as $\lambda \nearrow \infty$. A celebrated result of Avakumovic-Hörmander for compact boundaryless Riemannian manifolds (M^d, g) is that the Weyl asymptotics (asymptotic of $\mathscr{N}(\lambda; M^d)$ as $\lambda \nearrow \infty$) has the from,

$$\mathscr{N}(\lambda; M^d) = \frac{\mathrm{Vol}_g(M^d)\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{d-1}{2}}\right), \qquad \lambda \nearrow \infty, \tag{1.2.3}$$

where $\mathrm{Vol}_g(M^d)$ is the volume of M^d with respect to dv_g and ω_d is the volume of the unit d-ball in the Euclidean space $\mathbb{R}^d$, that is, $\omega_d = |\mathbb{B}_1^d|$. As a general statement it is known that the above asymptotics are sharp in the sense that the d-dimensional sphere $\mathbb{S}^d$ equipped with the usual round metric has precisely the same order remainder term as above. However, there are specific manifolds for which the asymptotic formula is far from sharp. It turns out that the geodesic flow associated to the Riemannian manifold (M^d, g) plays a decisive role in the sharpness of the Avakumovic-Hörmander result. Motivated by this and our aforementioned study of the geodesic length counting function we seek to answer the question of whether or not the above asymptotics is sharp in the case of the orthogonal or unitary groups. In our quest to answer this question we prove that the spectral counting function $\mathscr{N} = \mathscr{N}(\lambda; \mathbb{G})$ of the these groups $\mathbb{G}$ equipped with a bi-invariant metric g has the asymptotics:

$$\mathscr{N}(\lambda; \mathbb{G}) = \frac{\omega_d \mathrm{Vol}_g(\mathbb{G})}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{1}{2}[d-1-\varepsilon]}\right), \qquad \lambda \nearrow \infty, \tag{1.2.4}$$

where $\varepsilon = (n-1)/(n+1)$ and $n = \mathrm{rank}(\mathbb{G})$. Thus when $\mathrm{rank}(\mathbb{G}) \geq 2$ the Avakumovic-Hörmander-Weyl remainder term is not sharp and this is precisely when the geodesic flow of $\mathbb{G}$ *fails* to be periodic. Note that the rank of a Lie group $\mathbb{G}$ is the dimension of its maximal commutative subgroups, i.e. the dimension of its maximal tori. More interestingly we prove that when $\mathrm{rank}(\mathbb{G}) \geq 8$ the exponent of λ in the remainder term can be improved using number theoretical tools to $(d-2)/2$ which is sharp. This therefore provides some interesting examples of compact manifolds where the sharp term remainder is known and is of lower order than that given by the famous Avakumovic-Hörmander result, i.e. (1.2.3).

1.3 A note to the reader

This thesis consists of five main chapters each corresponding to a paper written by myself and Dr Ali Taheri which are either published, under review or to be submitted imminently at the present time of writing. Furthermore, due to the thesis submission being via paper style, each chapter is kept as close as possible to the actual paper form of the chapter. However, to be as consistent as possible throughout the thesis some notation has changed from that used in the published form of the papers. Moreover each chapter will contain its own abstract and introduction explaining the motivation and background to the work carried out in that respective chapter.

Chapter 2

Twist maps as energy minimisers in homotopy classes: symmetrisation and the coarea formula

Abstract

Let $\mathbf{X} = \mathbf{X}[a,b] = \{x : a < |x| < b\} \subset \mathbb{R}^n$ with $0 < a < b < \infty$ fixed be an open annulus and consider the energy functional,

$$\mathbb{F}[u;\mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2}\,dx,$$

over the space of admissible incompressible Sobolev maps

$$\mathcal{A}(\mathbf{X}) = \Big\{ u \in W^{1,2}(\mathbf{X},\mathbb{R}^n) : \det \nabla u = 1 \ \textit{a.e.} \text{ in } \mathbf{X} \text{ and } u|_{\partial \mathbf{X}} = \phi \Big\},$$

where ϕ is the identity map of $\overline{\mathbf{X}}$. Motivated by the earlier works [74, 75] in this paper we examine the *twist* maps as extremisers of $\mathbb{F}$ over $\mathcal{A}(\mathbf{X})$ and investigate their minimality properties by invoking the coarea formula and a symmetrisation argument. In the case $n = 2$ where $\mathcal{A}(\mathbf{X})$ is a union of infinitely many disjoint homotopy classes we establish the minimality of these extremising twists in their respective homotopy classes a result that then leads to the latter twists being L^1-local minimisers of $\mathbb{F}$ in $\mathcal{A}(\mathbf{X})$. We discuss variants and extensions to higher dimensions as well as to related energy functionals.

Contents

2.1 Introduction and preliminaries

Let $\mathbf{X} = \mathbf{X}[a, b] = \{(x_1, ..., x_n) : a < |x| < b\}$ with $0 < a < b < \infty$ fixed be an open annulus in $\mathbb{R}^n$ and consider the energy functional

$$\mathbb{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2} \, dx, \tag{2.1.1}$$

over the space of incompressible Sobolev maps,

$$\mathcal{A}(\mathbf{X}) = \Big\{ u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ in } \mathbf{X} \text{ and } u|_{\partial \mathbf{X}} = \phi \Big\}. \tag{2.1.2}$$

Here and in future ϕ denotes the identity map of $\overline{\mathbf{X}}$ and so the last condition in (2.1.2) means that $u \equiv x$ on $\partial \mathbf{X}$ in the sense of traces.

By a twist map u on $\mathbf{X} \subset \mathbb{R}^n$ we mean a continuous self-map of $\overline{\mathbf{X}}$ onto itself which agrees with the identity map ϕ on the boundary $\partial \mathbf{X}$ and has the specific spherical polar representation (*see* [75]-[77] for background and further results)

$$u : (r, \theta) \mapsto (r, \mathrm{Q}(r)\theta) \,, \qquad x \in \overline{\mathbf{X}}. \tag{2.1.3}$$

Here $r = |x|$ lies in $[a, b]$ and $\theta = x/|x|$ sits on $\mathbb{S}^{n-1}$ with $\mathrm{Q} \in \mathbf{C}([a, b], \mathbf{SO}(n))$ satisfying $\mathrm{Q}(a) = \mathrm{Q}(b) = \mathbf{I}_n$. Therefore, Q forms a closed loop in $\mathbf{SO}(n)$ based at $\mathbf{I}_n$ and for this in sequel we refer to Q as the twist *loop* associated with u. Also note that (2.1.3) in cartesian form can be written as

$$u : x \mapsto \mathrm{Q}(r)x = r\mathrm{Q}(r)\theta, \qquad x \in \overline{\mathbf{X}}. \tag{2.1.4}$$

Next subject to a differentiability assumption on the twist loop Q it can be verified that $u \in \mathcal{A}(\mathbf{X})$ with its $\mathbb{F}$ energy simplifying to

$$\begin{aligned}\mathbb{F}[\mathrm{Q}(r)x; \mathbf{X}] &= \frac{1}{2}\int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2}\,dx = \frac{1}{2}\int_{\mathbf{X}} \frac{|\nabla \mathrm{Q}(r)x|^2}{|x|^2}\,dx \\ &= \frac{n}{2}\int_{\mathbf{X}} \frac{dx}{|x|^2} + \frac{\omega_n}{2}\int_a^b |\dot{\mathrm{Q}}|^2 r^{n-1}\,dr,\end{aligned} \tag{2.1.5}$$

where the last equality uses $|\nabla[\mathrm{Q}(r)x]|^2 = n + r^2|\dot{\mathrm{Q}}\theta|^2$. Now as the primary task here is to search for extremising twist maps we first look at the Euler-Lagrange equation associated with the loop energy $\mathbb{E} = \mathbb{E}[\mathrm{Q}]$ defined by the last integral in (2.1.5) over the loop space $\{\mathrm{Q} \in W^{1,2}([a,b]; \mathbf{SO}(n)) : \mathrm{Q}(a) = \mathrm{Q}(b) = \mathbf{I}_n\}$. Indeed this can be shown to take the form (see below for justification)

$$\frac{d}{dr}\left[\left(r^{n-1}\dot{\mathrm{Q}}\right)\mathrm{Q}^t\right] = 0, \tag{2.1.6}$$

with solutions $\mathrm{Q}(r) = \exp[-\beta(r)A]P$, where $P \in \mathbf{SO}(n)$, $A \in \mathbb{R}^{n\times n}$ is skew-symmetric and $\beta = \beta(|x|)$ is described for $a \le r \le b$ by

$$\beta(r) = \begin{cases} \ln 1/r & n = 2, \\ r^{2-n}/(n-2) & n \ge 3. \end{cases} \tag{2.1.7}$$

Now to justify (2.1.6) fix $\mathrm{Q} \in W^{1,2}([a,b], \mathbf{SO}(n))$ and for $F \in W_0^{1,2}([a,b], \mathbb{R}^{n\times n})$ set $H = (F - F^t)\mathrm{Q}$ and $\mathrm{Q}_\epsilon = \mathrm{Q} + \epsilon H$. Then $\mathrm{Q}_\epsilon^t \mathrm{Q}_\epsilon = \mathbf{I}_n + \epsilon^2 H^t H$ and

$$\begin{aligned}\frac{d}{d\epsilon}\int_a^b 2^{-1}|\dot{\mathrm{Q}}_\epsilon|^2 r^{n-1}\,dr\Big|_{\epsilon=0} &= \int_a^b \langle \dot{\mathrm{Q}}, (\dot{F} - \dot{F}^t)\mathrm{Q} + (F - F^t)\dot{\mathrm{Q}}\rangle r^{n-1}\,dr \\ &= \int_a^b \langle \dot{\mathrm{Q}}, (\dot{F} - \dot{F}^t)\mathrm{Q}\rangle r^{n-1}\,dr \\ &= \int_a^b \langle \frac{d}{dr}(r^{n-1}\dot{\mathrm{Q}}\mathrm{Q}^t), (F - F^t)\rangle dr = 0,\end{aligned}$$

and so the arbitrariness of F with an orthogonality argument gives (2.1.6).

Returning to (2.1.1) it is not difficult to see that the Euler-Lagrange equation associated with $\mathbb{F}$ over $\mathcal{A}(\mathbf{X})$ is given by the system (*cf.* Section 2.4)

$$\frac{|\nabla u|^2}{|u|^4}u + div\left\{\frac{\nabla u}{|u|^2} - p(x)\mathrm{cof}\,\nabla u\right\} = 0, \qquad u = (u_1, ..., u_n), \tag{2.1.8}$$

where $p = p(x)$ is a suitable Lagrange multiplier. Here further analysis reveals that out of the solutions $\mathrm{Q} = \mathrm{Q}(r)$ to (2.1.6) just described only those twist loops in the form

$$\mathrm{Q}(r) = R\,\mathrm{diag}[\mathbf{R}[g](r), ..., \mathbf{R}[g](r)]R^t, \qquad R \in \mathbf{SO}(n), \tag{2.1.9}$$

when n is even and $\mathrm{Q}(r) \equiv \mathbf{I}_n$ $(a \le r \le b)$ when n is odd can grant extremising twist maps u for the original energy (2.1.1). For clarification $\mathbf{R}[g]$ denotes the $\mathbf{SO}(2)$ matrix of rotation by angle g:

$$\mathbf{R}[g] = \left[\begin{array}{cc} \cos g & \sin g \\ -\sin g & \cos g \end{array} \right].$$

Indeed direct computations give the *angle* of rotation $g = g(r)$ to be

$$g(r) = 2\pi k \frac{\log(r/a)}{\log(b/a)} + 2\pi m, \qquad k, m \in \mathbb{Z}, \tag{2.1.10}$$

when $n = 2$ and

$$g(r) = 2\pi k \frac{(r/a)^{2-n} - 1}{(b/a)^{2-n} - 1} + 2\pi m, \qquad k, m \in \mathbb{Z}, \tag{2.1.11}$$

when $n \ge 4$ even. (See also [65], [75], [77] for complementing and further results.)

Our point of departure is (2.1.4)-(2.1.9) and the aim is to study the minimising properties the twist maps calculated above. Of particular interest is the case $n = 2$ where the space $\mathcal{A}(\mathbf{X})$ admits multiple homotopy classes $(\mathscr{A}_k : k \in \mathbb{Z})$. Here direct minimisation of the energy over these classes gives rise to a scale of associated minimisers (u_k). Using a symmetrisation argument and the coarea formula we show that the twist maps u_k with twist angle g as presented in (2.1.10) are indeed energy minimisers in $\mathscr{A}_k$ and as a result also L^1 local minimisers of $\mathbb{F}$ over $\mathcal{A}(\mathbf{X})$. We discuss variants and extensions including a larger scale of energies where similar techniques can be applied to establish minimality properties in homotopy classes.

2.2 The homotopy structure of the space of self-maps $\mathscr{C} = \mathscr{C}(\mathbf{X})$

The rich homotopy structure of the space of continuous self-maps of the annulus $\mathbf{X} = \mathbf{X}[a, b] \subset \mathbb{R}^n$ will prove useful later on in constructing local energy minimisers. For this reason here we give a quick outline of the main tools and results and refer the reader to [74] for further details and proofs. To this end set $\mathscr{C} = \mathscr{C}(\mathbf{X}) = \{f \in \mathbf{C}(\overline{\mathbf{X}}, \overline{\mathbf{X}}) : f = \phi \text{ on } \partial\mathbf{X}\}$ equipped with the uniform topology. A pair of maps $f_0, f_1 \in \mathscr{C}$ are homotopic *iff* there exists $H \in \mathbf{C}([0, 1] \times \overline{\mathbf{X}}; \overline{\mathbf{X}})$ such that, firstly, $H(0, x) = f_0(x)$ for all $\mathbf{x} \in \overline{\mathbf{X}}$, secondly, $H(1, x) = f_1(x)$ for all $\mathbf{x} \in \overline{\mathbf{X}}$ and finally $H(t, x) = x$ for all $t \in [0, 1]$, $x \in \partial\mathbf{X}$. The equivalence class consisting of all $g \in \mathscr{C}$ homotopic to a given $f \in \mathscr{C}$ is referred to as the homotopy class of f and is denoted by $[f]$. Now the homotopy classes $\{[f] : f \in \mathscr{C}\}$ can be characterised as follows depending on whether $n = 2$ or $n \ge 3$.

- $(n=2)$ Using polar coordinates, for $f \in \mathscr{C}$ and for $\theta \in [0, 2\pi]$ (*fixed*), the $\mathbb{S}^1$-valued curve γ_θ defined by

$$\gamma_\theta : [a,b] \to \mathbb{S}^1 \subset \mathbb{R}^2, \qquad \gamma_\theta : r \mapsto f|f|^{-1}(r,\theta),$$

has a well-defined index or winding number about the origin. Furthermore, due to continuity of f, this index is independent of the particular choice of $\theta \in [0, 2\pi]$. This assignment of an integer (or index) to a map $f \in \mathscr{C}$ will be denoted by

$$f \mapsto \mathbf{deg}(f|f|^{-1}). \tag{2.2.1}$$

Note firstly that this integer also agrees with the Brouwer *degree* of the map resulting from identifying $\mathbb{S}^1 \cong [a,b]/\{a,b\}$, justified as a result of $\gamma_\theta(a) = \gamma_\theta(b)$ and secondly that for a differentiable curve (taking advantage of the embedding $\mathbb{S}^1 \subset \mathbf{C}$) we have the explicit formulation

$$\mathbf{deg}(f|f|^{-1}) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}. \tag{2.2.2}$$

Proposition 2.2.1. $(n=2)$. *The map* $\mathbf{deg} : \{[f] : f \in \mathscr{C}\} \to \mathbb{Z}$ *is bijective. Moreover, for any pair of maps* $f_0, f_1 \in \mathscr{C}$*, we have*

$$[f_0] = [f_1] \iff \mathbf{deg}(f_0|f_0|^{-1}) = \mathbf{deg}(f_1|f_1|^{-1}). \tag{2.2.3}$$

- $(n \geq 3)$ Using the identification $\overline{\mathbf{X}} \cong [a,b] \times \mathbb{S}^m$ where for ease of notation we have set $m = n-1$ it is plain that for $f \in \mathscr{C}$ the map (Here as usual ϕ denotes the *identity* map of the m-sphere and $\mathbf{C}_\phi(\mathbb{S}^m, \mathbb{S}^m)$ is the path-connected component of $\mathbf{C}(\mathbb{S}^m, \mathbb{S}^m)$ containing ϕ.)

$$\omega : [a,b] \to \mathbf{C}_\phi(\mathbb{S}^m, \mathbb{S}^m), \qquad \omega : r \mapsto f|f|^{-1}(r,\cdot),$$

uniquely defines an element of the fundamental group $\pi_1[\mathbf{C}_\phi(\mathbb{S}^m, \mathbb{S}^m)]$. By considering the action of $\mathbf{SO}(n)$ on $\mathbb{S}^m$ – viewed as its group of orientation *preserving* isometries, i.e., through the assignment,

$$\mathbf{E} : \xi \in \mathbf{SO}(n) \mapsto \omega \in \mathbf{C}(\mathbb{S}^m, \mathbb{S}^m), \tag{2.2.4}$$

where

$$\omega(x) = \mathbf{E}[\xi](x) = \xi x, \qquad x \in \mathbb{S}^m, \tag{2.2.5}$$

it can be proved that the latter assignment induces a group isomorphism on the level of the fundamental groups, namely,

$$\mathbf{E}^\star : \pi_1[\mathbf{SO}(n), I_n] \cong \pi_1[\mathbf{C}_\phi(\mathbb{S}^m, \mathbb{S}^m), \phi] \cong \mathbb{Z}_2. \tag{2.2.6}$$

Thus, summarising, we are naturally lead to the assignment of an integer mod 2 to any $f \in \mathscr{C}$ which will be denoted by

$$f \mapsto \mathbf{deg}_2(f|f|^{-1}) \in \mathbb{Z}_2. \tag{2.2.7}$$

Proposition 2.2.2. ($n \geq 3$) *The degree mod 2 map* $\mathbf{deg}_2 : \{[f] : f \in \mathscr{C}\} \to \mathbb{Z}_2$ *is bijective. Moreover, for a pair of maps* $f_0, f_1 \in \mathscr{C}$, *we have*

$$[f_0] = [f_1] \iff \mathbf{deg}_2(f_0|f_0|^{-1}) = \mathbf{deg}_2(f_1|f_1|^{-1}). \tag{2.2.8}$$

2.3 A countable family of L^1 local minimisers of $\mathbb{F}$ when $n = 2$

When $n = 2$ by Lebesgue monotonicity and degree theory (*see* [74] as well as [57],[61],[73],[36]) every map u in $\mathcal{A} = \mathcal{A}(\mathbf{X})$ has a representative (again denoted u) in $\mathscr{C}$. As a consequence we can introduce the components – hereafter called the homotopy classes,

$$\mathscr{A}_k := \left\{ u \in \mathcal{A} : \mathbf{deg}(u|u|^{-1}) = k \right\}, \qquad k \in \mathbb{Z}. \tag{2.3.1}$$

Evidently $\mathscr{A}_k$ are pairwise disjoint and their union (over all $k \in \mathbb{Z}$) gives $\mathcal{A}$. Furthermore it can be seen without difficulty that each $\mathscr{A}_k$ is $W^{1,2}$-sequentially weakly closed and that for $u \in \mathscr{A}_k$ and $s > 0$ there exists $\delta = \delta(u, s) > 0$ with

$$\{v : \mathbb{F}[v] < s\} \cap \mathbb{B}_\delta^{L^1}(u) \subset \mathscr{A}_k. \tag{2.3.2}$$

Here $\mathbb{B}_\delta^{L^1}(u) = \{v \in \mathcal{A} : ||v - u||_{L^1} < \delta\}$, that is, the L^1-ball in $\mathcal{A}$ centred at u. Indeed for the sequential weak closedness fix k and pick $(u_j : j \geq 1) \subset \mathscr{A}_k$ so that $u_j \rightharpoonup u$ in $W^{1,2}$. Then by a classical result of Y. Reshetnyak

$$\det \nabla u_j \overset{*}{\rightharpoonup} \det \nabla u \tag{2.3.3}$$

(as measures) and so $u \in \mathcal{A}$ while $u_j \to u$ uniformly on $\overline{\mathbf{X}}$ gives by Proposition 2.2.1 that $u \in \mathscr{A}_k$. For the second assertion arguing indirectly and assuming the contrary there exist $u \in \mathscr{A}_k$, $s > 0$ and $(v_j : j \geq 1)$ in $\mathcal{A}$ such that $\mathbb{F}[v_j; \mathbf{X}] < s$ and $||v_j - u||_{L^1} \to 0$ while $v_j \notin \mathscr{A}_k$. However by passing to a subsequence (not re-labeled) $v_j \rightharpoonup u$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ and as above $v_j \to u$ uniformly on $\overline{\mathbf{X}}$. Hence again by Proposition 2.2.1, $v_j \in \mathscr{A}_k$ for large enough j, which is a contradiction. $\square$

Now in view of the sequential weak lower semicontinuity of $\mathbb{F}$ in $\mathcal{A}$ (see below) an application of the direct methods of the calculus of variations leads to the following existence and multiplicity result.

Theorem 2.3.1. (*Local minimisers*) *Let $\mathbf{X} = \mathbf{X}[a,b] \subset \mathbb{R}^2$ and for $k \in \mathbb{Z}$ consider the homotopy classes $\mathscr{A}_k$ as defined by* (2.3.1)*. Then there exists $u = u(x;k) \in \mathscr{A}_k$ such that*

$$\mathbb{F}[u;\mathbf{X}] = \inf_{v\in\mathscr{A}_k} \mathbb{F}[v;\mathbf{X}]. \tag{2.3.4}$$

Furthermore for each such minimiser u there exists $\delta = \delta(u) > 0$ such that

$$\mathbb{F}[u;\mathbf{X}] \le \mathbb{F}[v;\mathbf{X}], \tag{2.3.5}$$

for all $v \in \mathcal{A}(\mathbf{X})$ satisfying $||u - v||_{L^1} < \delta$.

Proof. Fix k and pick $(v_j) \subset \mathscr{A}_k$ an *infimizing* sequence: $\mathbb{F}[v_j] \downarrow L := \inf_{\mathscr{A}_k} \mathbb{F}[\cdot]$. Then as $L < \infty$ and $a \le |v(x)| \le b$ for $v \in \mathcal{A}$ it follows that by passing to a subsequence (not re-labeled) $v_j \rightharpoonup u$ in $W^{1,2}(\mathbf{X},\mathbb{R}^2)$ and uniformly in $\overline{\mathbf{X}}$ where by the above discussion $u \in \mathscr{A}_k$. Now

$$\begin{aligned}\left|\int_{\mathbf{X}} \frac{|\nabla v_k|^2}{|v_k|^2} - \int_{\mathbf{X}} \frac{|\nabla v_k|^2}{|u|^2}\right| &\le \int_{\mathbf{X}} |\nabla v_k|^2 \left(\frac{||v_k|^2 - |u|^2|}{|u|^2|v_k|^2}\right)\\ &\le \sup_{\overline{\mathbf{X}}} \frac{||v_k|^2 - |u|^2|}{|u|^2|v_k|^2} \int_{\mathbf{X}} |\nabla v_k|^2 \to 0\end{aligned}$$

as $k \nearrow \infty$ together with

$$\int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2} \le \underline{\lim} \int_{\mathbf{X}} \frac{|\nabla v_k|^2}{|u|^2} \tag{2.3.6}$$

gives the desired lower semicontinuity of the $\mathbb{F}$ energy on $\mathcal{A}(\mathbf{X})$ as claimed, i.e.,

$$\mathbb{F}[u;\mathbf{X}] = \int_{\mathbf{X}} \frac{|\nabla u|^2}{2|u|^2} \le \underline{\lim} \int_{\mathbf{X}} \frac{|\nabla v_k|^2}{2|v_k|^2} = \underline{\lim}\, \mathbb{F}[v_k;\mathbf{X}]. \tag{2.3.7}$$

As a result $L \le \mathbb{F}[u] \le \liminf \mathbb{F}[v_j] \le L$ and so u is a minimiser as required. To justify the second assertion fix $k \in \mathbb{Z}$ and u as above and with $s = 1 + \mathbb{F}[u]$ pick $\delta > 0$ as in the discussion prior to the theorem. Then any $v \in \mathcal{A}$ satisfying $||u - v||_{L^1} < \delta$ also satisfies (2.3.5) [otherwise $\mathbb{F}[v] < \mathbb{F}[u] < s$ implying that $v \in \mathscr{A}_k$ and hence in view of u being a minimiser, $\mathbb{F}[v] \ge \mathbb{F}[u]$ which is a contradiction.] □

2.4 Twist maps and the Euler-Lagrange equation associated with $\mathbb{F}$

The purpose of this section is to formally derive the Euler-Lagrange equation associated with $\mathbb{F}$ over $\mathcal{A}(\mathbf{X})$. Note that $\mathbb{F}[u]$ can in principle be infinite if $|u|$ is too small or zero, however, for twist maps or more generally L^n-integrable maps in $\mathcal{A}$, $|u|$ is bounded away from zero as u is a self-map of $\overline{\mathbf{X}}$ onto itself. Moreover $\det \nabla u$ is L^1-integrable for the latter maps but not in general for maps u of Sobolev class $W^{1,2}$ (with $n \ge 3$).

Now the derivation uses the Lagrange multiplier method and proceeds formally by considering the unstrained functional

$$\mathbb{K}[u; \mathbb{X}] = \int_{\mathbf{X}} \left[\frac{|\nabla u|^2}{2|u|^2} - p(x) \left(\det \nabla u - 1 \right) \right] \, dx, \tag{2.4.1}$$

for suitable $p = p(x)$ where evidently $\mathbb{K}[u; \mathbf{X}] = \mathbb{F}[u; \mathbf{X}]$ when $u \in \mathcal{A}(\mathbf{X})$. We can calculate the first variation of this energy by setting $d/d\varepsilon \, \mathbb{K}[u_\varepsilon; \mathbf{X}]|_{\varepsilon=0} = 0$, where $u \in \mathcal{A}(\mathbf{X})$ is sufficiently regular and satisfies $|u| \geq c > 0$ in $\mathbf{X}$, $u_\varepsilon = u + \varepsilon\varphi$ for all $\varphi \in \mathbf{C}^\infty_c(\mathbf{X}, \mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ sufficiently small, hence obtaining

$$\begin{aligned}
0 =& \frac{d}{d\varepsilon} \int_{\mathbf{X}} \left[\frac{|\nabla u_\varepsilon|^2}{2|u_\varepsilon|^2} - p(x) \left(\det \nabla u_\varepsilon - 1 \right) \right] \, dx \bigg|_{\varepsilon=0} \\
=& \int_{\mathbf{X}} \left\{ \sum_{i,j=1}^{n} \left[\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} - p(x)[\mathrm{cof} \nabla u]_{ij} \right] \frac{\partial \varphi_i}{\partial x_j} - \sum_{i=1}^{n} \frac{|\nabla u|^2}{|u|^4} u_i \varphi_i \right\} dx \\
=& \int_{\mathbf{X}} \left\{ - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left[\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} - p(x)[\mathrm{cof} \nabla u]_{ij} \right] \varphi_i - \sum_{i=1}^{n} \frac{|\nabla u|^2}{|u|^4} u_i \varphi_i \right\} dx \\
=& \int_{\mathbf{X}} - \sum_{i=1}^{n} \left\{ \frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} - p(x)[\mathrm{cof} \nabla u]_{ij} \right] \right\} \varphi_i \, dx.
\end{aligned}$$

As this is true for every compactly supported φ as above an application of the fundamental lemma of the calculus of variation results in the Euler-Lagrange system for $u = (u_1, ..., u_n)$ in $\mathbf{X}$:

$$\frac{|\nabla u|^2}{|u|^4} u + div \left\{ \frac{\nabla u}{|u|^2} - p(x) \mathrm{cof} \, \nabla u \right\} = 0, \tag{2.4.2}$$

where the divergence operator is taken row-wise. Proceeding further for suitably regular u an application of the Piola identity on the cofactor term gives (with $1 \leq i \leq n$)

$$\frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \left\{ \frac{\partial}{\partial x_j} \left(\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} \right) - [\mathrm{cof} \, \nabla u]_{ij} \frac{\partial p}{\partial x_j} \right\} = 0. \tag{2.4.3}$$

Next expanding the differentiation further allows us to write

$$\begin{aligned}
0 =& \frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \left\{ \frac{1}{|u|^2} \frac{\partial^2 u_i}{\partial x_j^2} - \frac{2}{|u|^4} \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_j} u_k - [\mathrm{cof} \, \nabla u]_{ij} \frac{\partial p}{\partial x_j} \right\} \\
=& \frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \left\{ \frac{1}{|u|^2} \frac{\partial^2 u_i}{\partial x_j^2} - \frac{2}{|u|^4} \frac{\partial u_i}{\partial x_j} [\nabla u^t u]_j - [\mathrm{cof} \, \nabla u]_{ij} \frac{\partial p}{\partial x_j} \right\}.
\end{aligned} \tag{2.4.4}$$

Finally transferring back into vector notation and invoking the incompressibilty condition $\det \nabla u = 1$ it follows in turn that

$$\frac{\Delta u}{|u|^2} + \frac{|\nabla u|^2}{|u|^4} u - \frac{2}{|u|^4} \nabla u (\nabla u)^t u = (\mathrm{cof} \, \nabla u) \nabla p, \tag{2.4.5}$$

and subsequently

$$\frac{(\nabla u)^t}{|u|^2}\left[\Delta u + \frac{|\nabla u|^2}{|u|^2}u - \frac{2}{|u|^2}\nabla u(\nabla u)^t u\right] = \nabla p. \tag{2.4.6}$$

Thus the Euler-Lagrange system (2.4.2) is equivalent to (2.4.6) that in particular asks for the nonlinear term on the left of (2.4.6) to be a gradient field in $\mathbf{X}$. Recall from earlier discussion that restricting $\mathbb{F}$ to the class of twist maps results in the Euler-Lagrange equation (2.1.6) where the solution $\mathrm{Q} = \mathrm{Q}(r)$ as explicitly computed is the twist loop associated with the map

$$u : (r, \theta) \mapsto (r, \mathrm{Q}(r)\theta), \qquad x \in \mathbf{X}, \tag{2.4.7}$$

with $\mathrm{Q}(r) = \exp[-\beta(r)A]P$, $P \in \mathbf{SO}(n)$, $A \in \mathbb{R}^{n\times n}$ skew-symmetric ($A^t = -A$). The boundary condition $u = \phi$ on $\partial\mathbf{X}$ gives, (The function $\beta = \beta(r)$ was introduced earlier in Section 2.1.)

$$\exp[-\beta(a)A]P = \mathbf{I}_n, \qquad \exp[-\beta(b)A]P = \mathbf{I}_n. \tag{2.4.8}$$

Therefore it must be that $P = \exp[\beta(a)A]$ and $\exp([\beta(b) - \beta(a)]A) = \mathbf{I}_n$. Now as A lies in $\mathfrak{so}(n)$ it must be conjugate to a matrix S in the Lie algebra of the standard maximal torus of orthogonal 2-plane rotations in $\mathbf{SO}(n)$. This means that there exists $R \in \mathbf{SO}(n)$ such that $A = RSR^T$ for some S as described and so $S \in (\beta(b) - \beta(a))^{-1}\mathbb{L}$ where $\mathbb{L} = \{T \in \mathfrak{t} : \exp(T) = \mathbf{I}_n\}$, that is, $\mathbb{L}$ is the lattice in the Lie subalgebra $\mathfrak{t} \subset \mathfrak{so}(n)$ consisting of matrices sent by the exponential map to the identity $\mathbf{I}_n$ of $\mathbf{SO}(n)$. Hence $\mathrm{Q}(r) = R\exp(-[\beta(r) - \beta(a)]S)R^t$. Next upon noting that the derivatives of $\beta = \beta(r)$ are given by

$$\dot\beta(r) = -\frac{1}{r^{n-1}}, \qquad \ddot\beta(r) = \frac{n-1}{r^n},$$

we can write

$$\dot{\mathrm{Q}} = \frac{A\mathrm{Q}}{r^{n-1}}, \qquad \ddot{\mathrm{Q}} = \frac{A^2\mathrm{Q}}{r^{2n-2}} - (n-1)\frac{A\mathrm{Q}}{r^n}. \tag{2.4.9}$$

Now, moving forward, a set of straightforward calculations show that for a twist map u with a twice continuously differentiable twist loop $\mathrm{Q} = \mathrm{Q}(r)$ we have the differential relations

$$\begin{aligned}
&(\nabla u)^t = \mathrm{Q}^t + r\theta \otimes \dot{\mathrm{Q}}\theta, \\
&|\nabla u|^2 = tr[(\nabla u)^t(\nabla u)] = n + r^2|\dot{\mathrm{Q}}\theta|^2,
\end{aligned} \tag{2.4.10}$$

and likewise

$$\Delta u = \left[(n+1)\dot{\mathrm{Q}} + r\ddot{\mathrm{Q}}\right]\theta. \tag{2.4.11}$$

Thus for the particular choice of a twist map with twist loop arising from a solution to (2.1.6) the above quantities can be explicitly described by the relations

$$(\nabla u)^t = \mathrm{Q}^t + r^{2-n}\theta \otimes A\mathrm{Q}\theta, \tag{2.4.12}$$

$$\begin{aligned} |\nabla u|^2 &= tr[(\mathrm{Q}^t + r^{2-n}\theta \otimes A\mathrm{Q}\theta)(\mathrm{Q} + r^{2-n}A\mathrm{Q}\theta \otimes \theta)] \\ &= n + \frac{|A\mathrm{Q}\theta|^2}{r^{2(n-2)}}, \end{aligned} \tag{2.4.13}$$

and likewise

$$\begin{aligned} \Delta u &= \left[(n+1)\frac{A\mathrm{Q}}{r^{n-1}} + r\left(\frac{A^2\mathrm{Q}}{r^{2n-2}} - (n-1)\frac{A\mathrm{Q}}{r^n}\right)\right]\theta \\ &= \left[\frac{2A\mathrm{Q}}{r^{n-1}} + \frac{A^2Q}{r^{2n-3}}\right]\theta. \end{aligned} \tag{2.4.14}$$

For the ease of notation we shall hereafter write $\omega = \mathrm{Q}\theta$. Proceeding now with the calculations and using (2.4.13)-(2.4.14) we have

$$\Delta u + \frac{|\nabla u|^2}{|u|^2}u = \left[2\frac{A}{r^{n-1}} + \frac{A^2}{r^{2n-3}} + \frac{1}{r}\left(n + \frac{|A\omega|^2}{r^{2n-4}}\right)\mathbf{I}_n\right]\omega \tag{2.4.15}$$

and in a similar way

$$\begin{aligned} \frac{\nabla u(\nabla u)^t}{|u|^2}u &= \left[\mathrm{Q} + r^{2-n}A\omega \otimes \theta\right]\left[\mathrm{Q}^t + r^{2-n}\theta \otimes A\omega\right]\frac{\omega}{r} \\ &= \left[\mathbf{I}_n + \frac{A\omega \otimes \theta\mathrm{Q}^t + \mathrm{Q}\theta \otimes A\omega}{r^{n-2}} + \frac{A\omega \otimes A\omega}{r^{2n-4}}\right]\frac{\omega}{r} \\ &= \left[\mathbf{I}_n + \frac{A\omega \otimes \omega + \omega \otimes A\omega}{r^{n-2}} + \frac{A\omega \otimes A\omega}{r^{2n-4}}\right]\frac{\omega}{r}. \\ &= \frac{1}{r}(\mathbf{I}_n + r^{2-n}A)\omega, \end{aligned} \tag{2.4.16}$$

where the last identity here uses $(x \otimes y)z = \langle y, z\rangle x$ and $\langle \omega, \omega\rangle = \langle \mathrm{Q}\theta, \mathrm{Q}\theta\rangle = 1$ along with $\langle A\omega, \omega\rangle = 0$ for skew-symmetric A. Hence putting together (2.4.15) and (2.4.16) gives

$$\Delta u + \frac{|\nabla u|^2}{|u|^2}u - 2\frac{\nabla u(\nabla u)^t}{|u|^2}u = \left[\frac{A^2 + |A\omega|^2\mathbf{I}_n}{r^{2n-2}} + (n-2)\mathbf{I}_n\right]\frac{\omega}{r}, \tag{2.4.17}$$

which when combined with (2.4.12) results in

$$\begin{aligned} (2.4.6) &= \frac{(\nabla u)^t}{|u|^2}\left[\Delta u + \frac{|\nabla u|^2}{|u|^2}u - 2\frac{\nabla u(\nabla u)^t}{|u|^2}u\right] \\ &= \frac{1}{r^2}\left[\mathrm{Q}^t + \frac{\theta \otimes A\omega}{r^{n-2}}\right]\left[\frac{A^2 + |A\omega|^2\mathbf{I}_n}{r^{2n-4}} + (n-2)\mathbf{I}_n\right]\frac{\omega}{r} \\ &= \mathrm{Q}^t\left[\frac{A^2 + |A\omega|^2\mathbf{I}_n}{r^{2n-1}} + \frac{n-2}{r^3}\mathbf{I}_n\right]\omega + \frac{(\theta \otimes A\omega)A^2\omega}{r^{3n-3}}. \end{aligned} \tag{2.4.18}$$

Noting $(\theta \otimes A\omega)A^2\omega = \langle A\omega, A^2\omega\rangle\theta = \langle \mu, A\mu\rangle = 0$ with A skew-symmetric and $\mu = A\omega$ the last set of equations give

$$
\begin{aligned}
(2.4.6) &= \frac{(\nabla u)^t}{|u|^2}\left[\Delta u + \frac{|\nabla u|^2}{|u|^2}u - 2\frac{\nabla u(\nabla u)^t}{|u|^2}u\right] \\
&= \mathrm{Q}^t\left[\frac{A^2 + |A\omega|^2\mathbf{I}_n}{r^{2n-2}} + \frac{n-2}{r^2}\mathbf{I}_n\right]\frac{\omega}{r} = \mathcal{I}.
\end{aligned} \tag{2.4.19}
$$

Therefore to see if (2.4.6) admits twist solutions it suffices to verify if the quantity described by (2.4.19) is a gradient field in $\mathbf{X}$. Towards this end recall that here we have $\mathrm{Q}(r) = \exp(-\beta(r)A)P$ where as seen $P = \exp(\beta(a)A)$. Thus a basic calculation gives

$$
\begin{aligned}
|A\omega|^2 = |A\mathrm{Q}\theta|^2 &= \theta^t\mathrm{Q}^tA^tA\mathrm{Q}\theta = -\theta^t\mathrm{Q}^tA^2\mathrm{Q}\theta \\
&= -\theta^tP^t\exp(\beta(r)A)A^2\exp(-\beta(r)A)P\theta \\
&= -\theta P^tA^2P\theta = \theta P^tA^tAP\theta \\
&= |AP\theta|^2,
\end{aligned}
$$

and likewise by substitution we have

$$
\mathrm{Q}^tA^2\omega = P^tA^2P\theta.
$$

Hence using the above we can proceed by writing the Euler-Lagrange equation (2.4.19) upon substitution as,

$$
\begin{aligned}
\mathcal{I} &= \frac{(\nabla u)^t}{|u|^2}\left[\Delta u + \frac{|\nabla u|^2}{|u|^2}u - 2\frac{\nabla u(\nabla u)^t}{|u|^2}u\right] \\
&= P^t\left(A^2 + |AP\theta|^2I\right)P\frac{\theta}{r^{2n-1}} + (n-2)\frac{\theta}{r^3}.
\end{aligned} \tag{2.4.20}
$$

Now as for a fixed skew-symmetric matrix B by basic differentiation we have

$$
\nabla\left(|By|^2\right) = -2B^2y, \qquad \nabla|y|^{2n} = 2n|y|^{2n-2}y,
$$

it is evident that we can write

$$
-\nabla\left(\frac{|By|^2}{2n|y|^{2n}}\right) = \frac{B^2y}{n|y|^{2n}} + \frac{|By|^2y}{|y|^{2n+2}}. \tag{2.4.21}
$$

In particular with $B = P^tAP$ being skew-symmetric, (2.4.20) can be written in the form

$$
\begin{aligned}
\mathcal{I} &= \frac{(\nabla u)^t}{|u|^2}\left[\Delta u + \frac{|\nabla u|^2}{|u|^2}u - 2\frac{\nabla u(\nabla u)^t}{|u|^2}u\right] \\
&= -\nabla\left(\frac{|P^tAPx|^2}{2n|x|^{2n}}\right) + (n-1)\frac{P^tA^2Px}{n|x|^{2n}} + -(n-2)\nabla\frac{1}{|x|}.
\end{aligned} \tag{2.4.22}
$$

Therefore it is plain that (2.4.22) is a gradient field in $\mathbf{X}$ provided that the term on the right and subsequently the middle term, that is, the expression

$$
(n-1)\frac{P^tA^2Px}{n|x|^{2n}} \tag{2.4.23}
$$

is a gradient field in $\mathbf{X}$. By direct calculations (*cf.* [64]) this is seen to be the case *iff* all the eigenvalues of the symmetric matrix A^tA are equal. (Note that in odd dimensions this requirement leads to $A = 0$.) As a result here (2.4.22) would be a gradient (indeed ∇p) and so the Euler-Lagrange system (2.4.6) is satisfied by the twist u.

Now using the representation $A = RSR^t$ for some $S \in (\beta(b) - \beta(a))^{-1}\mathbb{L}$ and writing $S = \lambda J$ where, J is the $n \times n$ block diagonal matrix: $J = 0$ when n is odd and $J = \mathrm{diag}\left(\mathbf{A}_1, \cdots, \mathbf{A}_{n/2}\right)$ when n is even, i.e.,

$$J = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_{n/2} \end{bmatrix} \qquad \mathbf{A}_j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.4.24}$$

it is required that $\lambda(\beta(b) - \beta(a))J \in \mathbb{L}$. But invoking the lattice structure of $\mathbb{L}$ this can happen *iff*

$$\lambda = \frac{2\pi k}{\beta(b) - \beta(a)}, \qquad k \in \mathbb{Z}, \tag{2.4.25}$$

and thus

$$u(x) = R \exp\left(-2k\pi \frac{\beta(r) - \beta(a)}{\beta(b) - \beta(a)} J\right) R^t x. \tag{2.4.26}$$

Noticing that here we have

$$\frac{\beta(r) - \beta(a)}{\beta(b) - \beta(a)} = \frac{\log(r/a)}{\log(b/a)}, \tag{2.4.27}$$

for $n = 2$ and

$$\frac{\beta(r) - \beta(a)}{\beta(b) - \beta(a)} = \frac{(r/a)^{2-n} - 1}{(b/a)^{2-n} - 1}, \tag{2.4.28}$$

for even $n \geq 4$ respectively we obtain the representation

$$u(x) = R \exp\left(-g(r)J\right) R^t x = \exp\left(-g(r)A\right) x, \tag{2.4.29}$$

where we have set $A = RJR^t$ and the angle of twist function $g = g(r)$ is given by (2.1.10) for $n = 2$ and (2.1.11) for even $n \geq 4$ respectively. For odd $n \geq 3$ as shown $A = 0$ and so the only twist solution to (2.4.6) is the trivial solution $u \equiv x$.

2.5 Symmetrisation as a means of energy reduction on $\mathcal{A}(\mathbf{X})$ when $n = 2$

Recall that the space of admissible maps $\mathcal{A}(\mathbf{X})$ consists of maps $u \in W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ satisfying the incompressibility condition $\det \nabla u = 1$ *a.e.* in $\mathbf{X}$ and $u|_{\partial \mathbf{X}} = \phi$. Also as mentioned earlier due to a Lebesgue-type monotonicity every such map is continuous on the closed annulus $\overline{\mathbf{X}}$ and using degree theory the image of the closed annulus is again the closed annulus itself; hence, the "*embedding*"

$$\mathcal{A}(\mathbf{X}) = \bigcup_{k \in \mathbb{Z}} \mathscr{A}_k \subset \mathscr{C}(\mathbf{X}), \tag{2.5.1}$$

where the components $\mathscr{A}_k$ here are as defined by (2.3.1). For the sake of future calculations it is useful to write (2.2.2) as

$$\mathbf{deg}(u|u|^{-1}) = \frac{1}{2\pi} \int_a^b \frac{u \times u_r}{|u|^2} \, dr = k \in \mathbb{Z}, \tag{2.5.2}$$

where $|x| = r$. (Note that we adopt the convention that in two dimensions the cross product is a scalar and not a vector.) When u is a twist map, specifically, $u = \mathrm{Q}[g]x$ the integral reduces to $g(b) - g(a) = 2\pi k$ where as before $g = g(r)$ is the angle of rotation function. We now proceed by reformulating the $\mathbb{F}$ energy of an admissible map $u \in \mathcal{A}(\mathbf{X})$ in a more suggestive way. Indeed switching to polar co-ordinates it is seen that

$$|\nabla u|^2 = |u_r|^2 + \frac{1}{r^2} |u_\theta|^2 \tag{2.5.3}$$

where

$$|u_r|^2 = \frac{(u \cdot u_r)^2 + (u \times u_r)^2}{|u|^2}, \qquad |u_\theta|^2 = \frac{(u \cdot u_\theta)^2 + (u \times u_\theta)^2}{|u|^2}.$$

Next we note that

$$(|u|_r)^2 = \frac{(u \cdot u_r)^2}{|u|^2}, \qquad (|u|_\theta)^2 = \frac{(u \cdot u_\theta)^2}{|u|^2}.$$

Hence the gradient term on the left in (2.5.3) can be expressed as

$$\begin{aligned} |\nabla u|^2 &= \frac{(u \cdot u_r)^2 + (u \times u_r)^2}{|u|^2} + \frac{(u \cdot u_\theta)^2 + (u \times u_\theta)^2}{r^2 |u|^2} \\ &= |\nabla |u||^2 + \frac{(u \times u_r)^2}{|u|^2} + \frac{(u \times u_\theta)^2}{r^2 |u|^2}. \end{aligned} \tag{2.5.4}$$

From this we therefore obtain the $\mathbb{F}$ energy as

$$\begin{aligned} \mathbb{F}[u; \mathbf{X}] &= \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2} \, dx = \frac{1}{2} \int_0^{2\pi} \int_a^b \frac{|\nabla u|^2}{|u|^2} r \, dr d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_a^b \left[\frac{|\nabla |u||^2}{|u|^2} + \frac{(u \times u_r)^2}{|u|^4} + \frac{(u \times u_\theta)^2}{r^2 |u|^4} \right] r \, dr d\theta. \end{aligned} \tag{2.5.5}$$

Let us first state the following useful identity that will be employed in obtaining a fragment of the lower bound on the energy: For $u \in \mathcal{A}(\mathbf{X})$ and *a.e.* $r \in [a, b]$,

$$\int_0^{2\pi} \frac{u(r,\theta) \times u_\theta(r,\theta)}{|u|^2} \, d\theta = 2\pi. \tag{2.5.6}$$

The proof of this identity is postponed until later on in Section 2.7 (*cf.* Proposition 2.7.3). Now assuming this for the moment an application of Jensen's inequality gives, again for *a.e.* $r \in [a,b]$,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(u\times u_\theta)^2}{|u|^4}\,d\theta \geq \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{u(r,\theta)\times u_\theta(r,\theta)}{|u|^2}\,d\theta\right)^2 = 1. \tag{2.5.7}$$

Hence it is plain that

$$\int_0^{2\pi}\int_a^b \frac{(u\times u_\theta)^2}{r^2|u|^4} r\,drd\theta \geq 2\pi\ln(b/a). \tag{2.5.8}$$

Therefore we have the following lower bound on the $\mathbb{F}$ energy of an admissible map u:

$$\mathbb{F}[u;\mathbf{X}] \geq \pi\ln(b/a) + \frac{1}{2}\int_0^{2\pi}\int_a^b\left[\frac{|\nabla|u||^2}{|u|^2} + \frac{(u\times u_r)^2}{|u|^4}\right] r\,drd\theta. \tag{2.5.9}$$

Interestingly here we have equality only for twist maps and so outside this class the inequality is strict (for more on questions of uniqueness *see* [60]). The next task is to show that by using a basic "*symmetrisation*" in $\mathcal{A}(\mathbf{X})$ we can reduce the energy which will then be the main ingredient in the proof of the result.

Proposition 2.5.1. (*Symmetrisation*) *Let $u \in \mathcal{A}(\mathbf{X})$ be an admissible map and associated with u define the angle of rotation function $g = g(r)$ by setting*

$$g(r) = \frac{1}{2\pi}\int_a^r\int_0^{2\pi}\frac{u\times u_r}{|u|^2}\,d\theta dr, \qquad a \leq r \leq b. \tag{2.5.10}$$

Then the twist map defined by $\bar{u}(x) = \mathrm{Q}[g]x$ with $\mathrm{Q} = \mathrm{Q}[g] = \mathbf{R}[g]$ has a smaller $\mathbb{F}$ energy than the original map u, that is,

$$\mathbb{F}[\bar{u};\mathbf{X}] \leq \mathbb{F}[u;\mathbf{X}]. \tag{2.5.11}$$

Furthermore if $u \in \mathscr{A}_k$ then the symmetrised twist map $\bar{u}$ satisfies $\bar{u} \in \mathscr{A}_k$. Thus the homotopy classes $\mathscr{A}_k$ are invariant under symmetrisation.

Proof. Clearly the symmetrised twist map $\bar{u}$ is in the same homotopy class as u since by definition $g \in W^{1,2}[a,b]$ satisfies $g(a) = 0$ and

$$g(b) = \frac{1}{2\pi}\int_a^b\int_0^{2\pi}\frac{u\times u_r}{|u|^2}\,d\theta dr = 2\pi k. \tag{2.5.12}$$

Therefore $\bar{u} \in \mathscr{A}_k$ as a result of (2.5.2). Next the $\mathbb{F}$ energy of $\bar{u}$ satisfies the bound

$$\begin{aligned}
\mathbb{F}[\bar{u};\mathbf{X}] - 2\pi\log(b/a) &= \pi\int_a^b |\dot{\mathrm{Q}}(r)|^2 r\,dr \\
&= \pi\int_a^b |\dot{g}(r)|^2 r\,dr \\
&= \pi\int_a^b\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{u\times u_r}{|u|^2}d\theta\right]^2 r\,dr \\
&\leq \frac{1}{2}\int_0^{2\pi}\int_a^b\frac{(u\times u_r)^2}{|u|^4} r\,drd\theta
\end{aligned} \tag{2.5.13}$$

where the last line is a result of Jensen's inequality. Therefore by referring to (2.5.9) all that is left is to justify the inequality

$$2\pi \log(b/a) \leq \int_{\mathbf{X}} \frac{|\nabla |u||^2}{|u|^2} \, dx. \tag{2.5.14}$$

Towards this end we use the isoperimetric inequality in the context of sets of finite perimeter and the coarea formula in the context of Sobolev spaces: For real-valued f and non-negative Borel g:

$$\int_{\mathbf{X}} g(x)|\nabla f| \, dx = \int_{\mathbb{R}} \int_{\{f=t\}} g(x) \, d\mathcal{H}^1(x) \, dt. \tag{2.5.15}$$

(*See*, e.g., [17, 53].) Then upon taking $f = |u| \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$ and $g = 1/|u|^2$ this gives

$$\int_{\mathbf{X}} \frac{|\nabla |u||}{|u|^2} \, dx = \int_a^b \left(\int_{\{|u|=t\}} d\mathcal{H}^1 \right) \frac{dt}{t^2} = \int_a^b \mathcal{H}^1(\{|u| = t\}) \, \frac{dt}{t^2}. \tag{2.5.16}$$

Now since the level sets $E_t = \{x \in \mathbf{X} : |u(x)| \leq t\}$ and $F_t = \{x \in \mathbf{X} : |x| \leq t\}$ enclose the same area due to $\det \nabla u = 1$ *a.e.* and the identity boundary condition. In particular arguing as in Section 4.2 we know that $u(\overline{\mathbf{X}}) = \overline{\mathbf{X}}$ and

$$N(y, u, \mathbf{X}) = \sharp\{x \in \mathbf{X} : x \in u^{-1}(y)\} = 1, \tag{2.5.17}$$

for almost every $y \in \mathbf{X}$. Then from Theorem 5.34 on page 145 of [34] we know that for any measurable set $A \subset \mathbf{X}$ that,

$$\int_A v \circ u(x) \, dx = \int_{\mathbb{R}^n} v(y) N(y, u, A) \, dy, \tag{2.5.18}$$

for any $v \in L^\infty(\mathbb{R}^n)$. Then taking $A = \mathbf{X}$ and $v(x) = \chi_{F_t} = \chi_{\mathbb{B}_t \setminus \mathbb{B}_a}$ we obtain that,

$$|F_t| = \int_{\mathbf{X}} \chi_{F_t}(u(x)) \, dx = \int_{\mathbf{X}} \chi_{E_t}(x) dx = |E_t|. \tag{2.5.19}$$

Note above we have used the fact that for $x \in \mathbf{X}$, $u(x) \in F_t$ *iff* $x \in E_t$. Furthermore (we can consider u as extended by identity inside $\{|x| < a\}$) an application of the isoperimetric inequality gives $2\pi t = \mathcal{H}^1(\{|x| = t\}) = \mathcal{H}^1(\partial^\star F_t) \leq \mathcal{H}^1(\partial^\star E_t) = \mathcal{H}^1(\{|u| = t\})$ for *a.e.* $t \in [a, b]$ (*cf.*, e.g., [17, 30]). Thus substituting in (2.5.16) results in the lower bound

$$\begin{aligned} \int_{\mathbf{X}} \frac{|\nabla |u||}{|u|^2} \, dx &= \int_a^b \mathcal{H}^1(\{|u| = t\}) \, \frac{dt}{t^2} \\ &\geq \int_a^b \mathcal{H}^1(\{|x| = t\}) \, \frac{dt}{t^2} = \int_a^b 2\pi t \, \frac{dt}{t^2} = 2\pi \log(b/a). \end{aligned} \tag{2.5.20}$$

Finally we arrive at the conclusion by noting that u and ϕ have the same distribution function, that is, again as a result of the pointwise constraint $\det \nabla u = 1$ *a.e.* in $\mathbf{X}$ and the boundary condition: $\alpha_u(t) = |\{x \in \mathbf{X} : |u(x)| \geq t\}| = |\{x \in \mathbf{X} : |x| \geq t\}| = \alpha_\phi(t)$ and therefore

$$\int_{\mathbf{X}} \frac{dx}{|u|^2} = \int_a^\infty -2\alpha_u(t)\,\frac{dt}{t^3} + \frac{|\mathbf{X}|}{a^2} = \int_a^\infty -2\alpha_\phi(t)\,\frac{dt}{t^3} + \frac{|\mathbf{X}|}{a^2} = \int_{\mathbf{X}} \frac{dx}{|x|^2} = 2\pi\log(b/a).$$

Now putting all the above together, a final application of Hölder inequality gives,

$$\begin{aligned}
(2\pi\log(b/a))^2 &\leq \left(\int_{\mathbf{X}} \frac{|\nabla|u||}{|u|^2}\,dx\right)^2 \\
&\leq \int_{\mathbf{X}} \frac{|\nabla|u||^2}{|u|^2}\,dx \times \int_{\mathbf{X}} \frac{dx}{|u|^2} \\
&= 2\pi\log(\frac{b}{a}) \int_{\mathbf{X}} \frac{|\nabla|u||^2}{|u|^2}\,dx
\end{aligned} \tag{2.5.21}$$

and thus eventually we have

$$2\pi\log(b/a) \leq \int_{\mathbf{X}} \frac{|\nabla|u||^2}{|u|^2}\,dx$$

and so the conclusion follows. □

2.6 The $\mathbb{F}$ energy and connection with the distortion function

In this section we delve into the relationship between the energy functional $\mathbb{F}$ in (2.1.1) and the notions of distortion function and energy of geometric function theory. In particular we show that in two dimensions twist maps have minimum distortion among all incompressible Sobolev homeomorphisms of the annulus with identity boundary values in any given homotopy class. To fix notation and terminology let $U, V \subset \mathbb{R}^n$ be open sets and $f \in W^{1,1}_{loc}(U, V)$. Then f is said to have finite outer distortion *iff* there exists measurable function $K = K(x)$ with $1 \leq K(x) < \infty$ such that

$$|\nabla f(x)|^n \leq n^{n/2} K(x) \det \nabla f(x). \tag{2.6.1}$$

The smallest such K is called the *outer* distortion of f and denoted by $K_O(x, f)$. Note that here $|A| = \sqrt{\mathrm{tr} A^t A}$ is the Hilbert-Schmidt norm of the $n \times n$ matrix A. Naturally $1 \leq K_O(x, f) < \infty$ and it measures the deviation of f from being conformal. We also speak of the *inner* distortion function $K = K_I(x, f)$ defined by the quotient

$$K_I(x, f) = \frac{n^{-n/2}|\mathrm{cof}\,\nabla f|^n}{\det(\mathrm{cof}\,\nabla f)}, \tag{2.6.2}$$

when $\det \nabla f(x) \neq 0$ and $K_I(x, f) = 1$ otherwise. We define the *distortion* energy associated to the inner distortion $K_I(x, f)$ (2.6.2) by the integral

$$\mathbb{W}[f; U] = \int_U \frac{K_I(x, f)}{|x|^n}\,dx. \tag{2.6.3}$$

Related energies and more have been considered in [49] with close links to the work in [3]. The connection between the $\mathbb{F}$ energy and the distortion energy $\mathbb{W}$ (2.6.3) is implicit in the following result of T. Iwaniec, G. Martin, J. Onninen and K. Astala [3]. (See also [2], [49] and [50].)

Theorem 2.6.1. *Suppose $f \in W^{1,n}_{loc}(\mathbf{X}, \mathbf{X})$ is a homeomorphism with finite outer distortion. Assume K_I is L^1-integrable over $\mathbf{X}$. Then the inverse map $h = f^{-1} : \mathbf{X} \to \mathbf{X}$ lies in the Sobolev space $W^{1,n}(\mathbf{X}, \mathbf{X})$. Furthermore*

$$n^{-\frac{n}{2}} \int_{\mathbf{X}} \frac{|\nabla h(y)|^n}{|h(y)|^n}\, dy = \int_{\mathbf{X}} \frac{K_I(x, f)}{|x|^n}\, dx. \tag{2.6.4}$$

Proof. The first assertion is Theorem 10.4 pp. 22 of [3]. For the second assertion using definitions we have

$$n^{n/2} K_I(x, f) = \frac{|\mathrm{cof}\, \nabla f|^n}{\det(\mathrm{cof}\, \nabla f)} = |(\nabla f)^{-1}|^n \det \nabla f = |\nabla h(f)|^n \det \nabla f,$$

and the conclusion follows upon dividing by $|x|^n$ and integrating using the area formula. □

Now let $\mathcal{A}^n(\mathbf{X}) = \{u \in W^{1,n}(\mathbf{X}; \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ in } \mathbf{X} \text{ and } u|_{\partial \mathbf{X}} = \phi\}$. As in Section 2.2 each u in $\mathcal{A}^n(\mathbf{X})$ admits a representative in $\mathscr{C}(\mathbf{X})$. Now restricting to homeomorphisms $u \in \mathcal{A}^n(\mathbf{X})$ the above theorem gives $v = u^{-1} \in \mathcal{A}^n(\mathbf{X})$ and so by the incompressibility constraint

$$\int_{\mathbf{X}} \frac{K_I(x, u)}{|x|^n}\, dx = n^{-\frac{n}{2}} \int_{\mathbf{X}} \frac{|\mathrm{cof}\, \nabla u|^n}{|x|^n}\, dx = n^{-\frac{n}{2}} \int_{\mathbf{X}} \frac{|\nabla v|^n}{|v|^n}\, dx. \tag{2.6.5}$$

In the planar case this allows us to relate the distortion energy of a homeomorphism u, say, in $\mathscr{A}_k$ to the $\mathbb{F}$ energy of the inverse map $v = u^{-1}$ in $\mathscr{A}_{-k}$ through

$$\mathbb{F}[v; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla v(y)|^2}{|v(y)|^2}\, dy = \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla u(x)|^2}{|x|^2}\, dx = \int_{\mathbf{X}} \frac{K_I(x, u)}{|x|^2}\, dx = \mathbb{W}[u; \mathbf{X}]. \tag{2.6.6}$$

Therefore by showing that twist maps minimise the $\mathbb{F}$ energy within their homotopy classes we have implicitly shown that twist maps minimise the distortion energy within their respective homotopy classes of homeomorphisms (as the inverse of a twist map is a twist map in *opposite* direction and clearly twist maps are homeomorphisms of annuli onto themselves).

Theorem 2.6.2. *The distortion energy $\mathbb{W}$ has a minimiser $u = u(x; k)$ ($k \in \mathbb{Z}$) among all homeomorphisms within $\mathscr{A}_k$. The minimiser is a twist map of the form $u = \mathrm{Q}[g]x$ where $g(r) = 2\pi k \ln(r/a)/\ln(b/a)$. In particular the minimum energy is given by,*

$$\mathbb{W}[u; \mathbf{X}] = \int_{\mathbf{X}} \frac{K_I(x, u)}{|x|^2}\, dx = 2\pi \ln(b/a) + 4\pi^3 k^2 / \ln(b/a). \tag{2.6.7}$$

Proof. That $u = u_k$ minimises $\mathbb{W}$ amongst homeomorphisms in $\mathscr{A}_k$ is a result of $u_{-k} = (u_k)^{-1}$ minimising $\mathbb{F}$ over $\mathscr{A}_{-k}$ (Proposition 2.5.1) and (2.6.6). Indeed arguing indirectly assume there is a homeomorphism $v \in \mathscr{A}_k$: $\mathbb{W}[v; \mathbf{X}] < \mathbb{W}[u_k; \mathbf{X}]$. Then by (2.6.6), $\mathbb{F}[v^{-1}; \mathbf{X}] < \mathbb{F}[u_{-k}; \mathbf{X}]$ and this is a contradiction as v^{-1}, $u_{-k} \in \mathscr{A}_{-k}$ while $\mathbb{F}[u_{-k}; \mathbf{X}] = \inf_{\mathscr{A}_{-k}} \mathbb{F}$. We are thus left with the calculation of the $\mathbb{W}$ energy of $u = u_k$. To this end put $v = u^{-1}$:

$$\begin{aligned}\mathbb{F}[v; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla v|^2}{|v|^2} \, dx &= 2\pi \ln(b/a) + \pi \int_a^b r \dot{g}_{-k}(r)^2 \, dr \\ &= 2\pi \ln(b/a) + \frac{4\pi^3 k^2}{\ln(b/a)},\end{aligned} \tag{2.6.8}$$

and so a further reference to (2.6.6) completes the proof. $\square$

In the higher dimensions, i.e. $n > 2$, from Theorem 2.6.1 we can get an analogous identity to (2.6.6) for homeomorphisms u in $\mathcal{A}^n$. Indeed, with $v = u^{-1}$

$$n\mathbb{F}[v; \mathbf{X}] = \int_{\mathbf{X}} \frac{|\nabla v(y)|^n}{|v(y)|^n} \, dy = \int_{\mathbf{X}} \frac{|\operatorname{cof} \nabla u(x)|^n}{|x|^n} \, dx = n^{n/2} \mathbb{W}[u; \mathbf{X}], \tag{2.6.9}$$

where the energies $\mathbb{F} = \mathbb{F}_n$ and $\mathbb{W}$ are given by,

$$\mathbb{F}[v; \mathbf{X}] = \frac{1}{n} \int_{\mathbf{X}} \frac{|\nabla v(y)|^n}{|v(y)|^n} \, dy, \qquad \mathbb{W}[u; \mathbf{X}] = \int_{\mathbf{X}} \frac{K_I(x, u)}{|x|^n} \, dx. \tag{2.6.10}$$

Therefore again to find minimisers of $\mathbb{W}$ among homeomorphisms in $u \in \mathcal{A}^n(\mathbf{X})$ one can follows the lead of $n = 2$ and consider the energy $\mathbb{F}$ over $\mathcal{A}^n(\mathbf{X})$. It is straightforward to see that we have equality in (2.6.9) for twist maps $u \in \mathcal{A}^n(\mathbf{X})$ with the distortion energy of $u = \mathrm{Q}(r)x$ given by

$$\mathbb{W}[u; \mathbf{X}] = n^{-n/2} \int_{\mathbf{X}} \left(n|x|^{-2} + |\dot{\mathrm{Q}}\theta|^2 \right)^{n/2} \, dx. \tag{2.6.11}$$

Now restricting to the particular case of the twist map being (*cf.* [65])

$$u(x) = \exp\left(-g(r)J\right) x, \qquad x \in \mathbf{X}, \tag{2.6.12}$$

with J is as in Section 2.4 and $g \in W^{1,n}[a, b]$ the angle of rotation describing the twist. The corresponding distortion energy is

$$\begin{aligned}\mathbb{W}[u; \mathbf{X}] &= n^{-n/2} \int_{\mathbf{X}} \left(n|x|^{-2} + |\dot{g}|^2 \right)^{n/2} \, dx \\ &= \omega_n n^{-n/2} \int_a^b \left(nr^{-2} + |\dot{g}|^2 \right)^{n/2} r^{n-1} \, dr.\end{aligned}$$

Note that in higher dimensions (i.e., $n \geq 3$) as discussed earlier there are only two homotopy classes in $\mathcal{A}^n(\mathbf{X})$. A twist map $u = \mathrm{Q}(r)x$ lies in the non-trivial homotopy class of $\mathcal{A}^n(\mathbf{X})$ *iff* the twist loop $\mathrm{Q} = \mathrm{Q}(r) \in \mathbf{C}([a,b], \mathbf{SO}(n))$ based at $\mathbf{I}_n$ lifts to a non-closed path $R = R(r) \in \mathbf{C}([a,b], \mathbf{Spin}(n))$ connecting ± 1 in $\mathbf{Spin}(n)$ (*see* [76] for more.) (Note that $\mathbf{Spin}(n)$ is the universal cover of $\mathbf{SO}(n)$ and $\{\pm 1\} \subset \mathbf{Spin}(n)$ is the fibre over $\mathbf{I}_n$ under the covering map.)

Likewise a twist u of the form (2.6.12) lies in the non-trivial homotopy class of $\mathcal{A}^n(\mathbf{X})$ *iff* the angle of rotation function g satisfies $g(b) - g(a) = 2\pi k$ for some k odd. When k is even the twist map u lies in the trivial homotopy class of $\mathcal{A}(\mathbf{X})$. (The identity boundary conditions on u dictates that the angle of rotation function must satisfy $g(b) - g(a) = 2\pi k$ for some $k \in \mathbb{Z}$.)

2.7 Other variants of the Dirichlet energy

The goal of this section is to establish various energy bounds and identities, when $n = 2$, by invoking the regularity and the measure preserving constraints satisfied by the elements of $\mathcal{A}(\mathbf{X})$. These inequalities will ultimately lead to useful results for extremisers and minimisers of variants of the Dirichlet energy in homotopy classes of $\mathcal{A}(\mathbf{X})$. We begin with the following identity.

Proposition 2.7.1. *For $\Phi \in \mathbf{C}^1[a,b]$ the integral identity*

$$\int_{\mathbf{X}} \Phi(|u|)\, dx = \int_{\mathbf{X}} \Phi(|x|)\, dx \tag{2.7.1}$$

holds for all $u \in \mathcal{A}(\mathbf{X})$.

Proof. Denoting by $\alpha_u = \alpha_u(t)$ the distribution function of $|u|$ we can write using basic considerations and invoking the standard properties of distribution functions

$$\begin{aligned}
\int_{\mathbf{X}} \Phi(|u|)\, dx &= \int_{\mathbf{X}} \int_a^{|u|} \dot{\Phi}(t)\, dt dx + \Phi(a)|\mathbf{X}| = \int_a^b \dot{\Phi}(t)\alpha_u(t)\, dt + \Phi(a)|\mathbf{X}| \\
&= \int_a^b \dot{\Phi}(t)\alpha_x(t)\, dt + \Phi(a)|\mathbf{X}| = \int_{\mathbf{X}} \int_a^{|x|} \dot{\Phi}(t)\, dt dx + \Phi(a)|\mathbf{X}| \\
&= \int_{\mathbf{X}} \Phi(|x|)\, dx,
\end{aligned} \tag{2.7.2}$$

which is the required conclusion. Alternatively this is an application of Theorem 5.34 on page 145 in [34]. $\square$

We now collect a few more results which will be needed for the proof of our main theorem at the end of this section.

Proposition 2.7.2. *Let $\Phi \in \mathbf{C}^1[a,b]$ and pick $u \in \mathcal{A}(\mathbf{X})$. Consider the continuous closed curve $\gamma(\theta) = u(r,\theta)$ where $a < r < b$ is fixed. Then the integral identity*

$$\int_0^{2\pi} \Phi(|u|)^2 (u \times u_\theta)\, d\theta \bigg|_a^r = 2 \int_{\mathbf{X}_r} \left[|u|\Phi(|u|)\dot{\Phi}(|u|) + \Phi(|u|)^2 \right] dx, \tag{2.7.3}$$

holds for almost every $r \in (a,b)$, where $\mathbf{X}_r = \mathbf{X}[a,r] = \{x \in \mathbb{R}^2 : a < |x| < r\}$.

Proof. We shall justify the assertion first when u is a smooth diffeomorphism and then pass on to the general case by invoking a suitable approximation argument. Towards this end consider first the case where u is a smooth diffeomorphism with $u \equiv x$ on $\partial \mathbf{X}$. [Here u need not satisfy the incompressibility condition in $\mathcal{A}(\mathbf{X})$.] Let α denote the 1-form

$$\alpha = \Phi(|x|)^2 \left(x_1 dx_2 - x_2 dx_1 \right). \tag{2.7.4}$$

Then by a rudimentary calculation the pull-back of α under the $\mathbf{C}^\infty$ curve γ is given by

$$\gamma^* \alpha = \Phi(|u|)^2 (u \times u_\theta)\, d\theta. \tag{2.7.5}$$

Hence contour integration and basic considerations lead to the integral identity

$$\int_\gamma \alpha = \int_0^{2\pi} \Phi(|u|)^2 (u \times u_\theta)\, d\theta \bigg|_r. \tag{2.7.6}$$

Note that the $\mathbf{C}^\infty$ curve γ here is diffeomorphic to $\mathbb{S}^1$ and as such by the Jordan-Schöenflies theorem γ is the boundary of some bounded region $C_\gamma \subset \mathbb{R}^2$ diffeomorphic to the unit ball $\mathbb{B}_1$. In particular due to the boundary conditions on u we have that $a < |\gamma|(\theta) = |u|(r,\theta)$ when $a < r$ and so as a result $C_a^\gamma = \{x \in \mathbb{R}^2 : a < |x| < |\gamma|\} \subset \mathbf{X}$ with boundaries of $\partial \mathbb{B}_a$ and γ. Hence application of Stoke's theorem gives

$$\int_{C_a^\gamma} d\alpha = \int_{\partial C_a^\gamma} \alpha = \int_\gamma \alpha - \int_{|x|=a} \alpha = \int_0^{2\pi} \Phi(|u|)^2 u \times u_\theta \, d\theta \bigg|_a^r, \tag{2.7.7}$$

for all $r \in [a,b]$. Next again by a rudimentary calculation we obtain that the pull-back of $d\alpha$ is,

$$d\alpha = 2 \det \nabla u \left[|u|\Phi(|u|)\dot{\Phi}(|u|) + \Phi(|u|)^2 \right] dx_1 \wedge dx_2. \tag{2.7.8}$$

Hence from (2.7.7) and (2.7.8) it follows that for all $r \in [a,b]$,

$$2 \int_{\mathbf{X}_r} \left[|u|\Phi(|u|)\dot{\Phi}(|u|) + \Phi(|u|)^2 \right] \det \nabla u \, dx = \int_0^{2\pi} \Phi(|u|)^2 u \times u_\theta \, d\theta \bigg|_a^r. \tag{2.7.9}$$

Now pick an arbitrary u in $\mathcal{A}(\mathbf{X})$. By approximation, e.g., using Theorem 1.1 in [41] there is a sequence of $\mathbf{C}^\infty$ diffeomorphisms (v^k) so that $v^k - u \in W_0^{1,2}(\mathbf{X}, \mathbb{R}^2)$ with $v_k \to u$ uniformly on $\overline{\mathbf{X}}$ and strongly in $W^{1,2}$. (Note the condition that u satisfies the INV property in Theorem 1.1 of [41] is straightforward since our mappings satisfy $\det \nabla u = 1$ *a.e.* and $u \in W^{1,2}(\mathbf{X}, \mathbb{R}^2)$. However this INV property is proved later for a larger class of mappings containing $\mathcal{A}(\mathbf{X})$ in Proposition 5.6.2 of Chapter 5. Additionally the constraint that $u(\mathbf{X}) \cap u(\partial \mathbf{X}) = \emptyset$ is only needed to show that $u(\mathbf{X})$ is open which for our mappings is also known as u have L^1 integrable dilation *see* [50]. Alternatively we could extend our mappings by identity onto a larger annulus $\mathbf{X}_\varepsilon$ where $u(\mathbf{X}_\varepsilon) \cap u(\partial \mathbf{X}_\varepsilon) = \emptyset$ will hold. Then proving (2.7.3) for the extened mappings on $\mathbf{X}_\varepsilon$ will lead to the result for u on $\mathbf{X}$ as the extended mappings are identity off of $\mathbf{X}$.) Hence,

$$f_k := \Phi(|v^k|)^2 \frac{v^k \times v^k_\theta}{|x|} \to \Phi(|u|)^2 \frac{u \times u_\theta}{|x|} =: f, \tag{2.7.10}$$

a.e. in $\mathbf{X}$. Note that $|f_k| \le c|v^k_\theta|$, $|f| \le c|u_\theta|$ for some $c > 0$ and so $f_k, f \in L^2(\mathbf{X})$ and by virtue of $v_k \to u$ in $W^{1,2}$ and dominated convergence, for each $r \in (a, b)$ and $0 < \delta < b - r$, we have

$$\int_r^{r+\delta} \int_0^{2\pi} \Phi(|v^k|)^2 (v^k \times v^k_\theta) \, d\theta dr \to \int_r^{r+\delta} \int_0^{2\pi} \Phi(|u|)^2 (u \times u_\theta) \, d\theta dr. \tag{2.7.11}$$

In a similar spirit we have (suppressing the arguments of Φ for brevity)

$$h_k := \left[|v^k| \Phi \dot{\Phi} + \Phi^2 \right] \det \nabla v^k \to \left[|u| \Phi \dot{\Phi} + \Phi^2 \right] \det \nabla u =: h, \tag{2.7.12}$$

a.e. in $\mathbf{X}$. Again since $|h_k| \le c|v^k_{x_1}||v^k_{x_2}|$, $|h| \le c|u_{x_1}||u_{x_2}|$ for some $c > 0$ we have $h_k, h \in L^1(\mathbf{X})$ and so by dominated convergence

$$\int_{\mathbf{X}_r} \left[|v^k| \Phi \dot{\Phi} + \Phi^2 \right] \det \nabla v^k \, dx \to \int_{\mathbf{X}_r} \left[|u| \Phi \dot{\Phi} + \Phi^2 \right] \det \nabla u \, dx. \tag{2.7.13}$$

Now combining (2.7.9) and (2.7.11) together with the fact that $u = v^k = \phi$ on $\partial \mathbf{X}$ it follows that

$$2 \int_r^{r+\delta} \int_{\mathbf{X}_r} \left[|v^k| \Phi \dot{\Phi} + \Phi^2 \right] \det \nabla v^k \, dx dr \to \int_r^{r+\delta} \int_0^{2\pi} \Phi(|u|)^2 (u \times u_\theta) \, d\theta \Big|_a^r \, dr.$$

Moreover (2.7.13) and a final application of dominated convergence gives

$$2 \int_r^{r+\delta} \int_{\mathbf{X}_r} \left[|u| \Phi \dot{\Phi} + \Phi^2 \right] \det \nabla u \, dx dr = \int_r^{r+\delta} \int_0^{2\pi} \Phi(|u|)^2 (u \times u_\theta) \, d\theta \Big|_a^r \, dr.$$

Therefore the result follows by recalling that $\det \nabla u = 1$ *a.e.* in $\mathbf{X}$ and applying the Lebsegue differentiation theorem, i.e., dividing by δ and letting $\delta \searrow 0$. □

Note that we can write the conclusion of the above proposition, namely, the integral identity (2.7.3) in a shorter and somewhat more suggestive form

$$\int_0^{2\pi} \Phi(|u|)^2 (u \times u_\theta) \, d\theta \Big|_a^r = \int_{\mathbf{X}_r} |u|^{-1} \dot{\Gamma}(|u|) \, dx, \tag{2.7.14}$$

where $\Gamma(t) = t^2\Phi(t)^2$ for $\Phi \in \mathbf{C}^1[a,b]$. With this result and formulation at our disposal we can now prove the earlier relation (2.5.6) as a specific proposition.

Proposition 2.7.3. *Taking $\Phi(t) = 1/t$ in the above gives the integral identity*

$$\int_0^{2\pi} \frac{u(r,\theta) \times u_\theta(r,\theta)}{|u|^2}\, d\theta = 2\pi, \qquad a.e.\ r \in [a,b]. \tag{2.7.15}$$

Proof. When $\Phi(t) = 1/t$ it can be easily seen that $\dot{\Gamma}(t) = 0$ and therefore Proposition 2.7.2 gives that,

$$\int_0^{2\pi} \frac{(u \times u_\theta)}{|u|^2}\, d\theta \Bigg|_a^r = 0, \quad a.e.\ r \in [a,b]. \tag{2.7.16}$$

Then recalling that $u(x) = x$ on $\partial\mathbf{X}$, i.e. when $|x| = a$, it can be seen that the integral over the inner boundary is 2π and hence from (2.7.16) this implies that,

$$\int_0^{2\pi} \frac{(u \times u_\theta)}{|u|^2}\, d\theta = 2\pi, \quad a.e.\ r \in [a,b], \tag{2.7.17}$$

which completes the proof. $\square$

Proposition 2.7.4. *Suppose $\Gamma \in \mathbf{C}^2[a,b]$ such that $\dot{\Gamma}(t)/t$ is a monotone increasing function. Then for almost every $r \in [a,b]$ we have that*

$$\int_0^{2\pi} \Gamma(|u|)^2 \frac{(u(r,\theta) \times u_\theta(r,\theta))^2}{|u|^4}\, d\theta \geq 2\pi\Gamma(r)^2. \tag{2.7.18}$$

Proof. First we note that by Proposition 2.7.2 we have for $a.e.\ r \in [a,b]$ that

$$\int_0^{2\pi} \Gamma(|u|) \frac{u \times u_\theta}{|u|^2}\, d\theta \Bigg|_a^r = \int_{\mathbf{X}_r} |u|^{-1}\dot{\Gamma}(|u|)dx. \tag{2.7.19}$$

As $\dot{\Gamma}(t)/t$ is monotone increasing and $|u|$ and $|x|$ share the same distribution function $\alpha_x(t)$ we have that,

$$\begin{aligned}
\int_{\mathbf{X}_r^b} |u|^{-1}\dot{\Gamma}(|u|)\, dx &= \int_a^b \frac{d}{dt}\left(\frac{\dot{\Gamma}(t)}{t}\right) \int_{\mathbf{X}} \chi_{\{x \in \mathbf{X}_r^b : |u|(x) > t\}}\, dx\, dt + |\mathbf{X}_r^b| \frac{\dot{\Gamma}(a)}{a} \\
&\leq \int_a^b \alpha_x(t) \frac{d}{dt}\left(\frac{\dot{\Gamma}(t)}{t}\right) dt + |\mathbf{X}_r^b| \frac{\dot{\Gamma}(a)}{a} \\
&= \int_{\mathbf{X}_r^b} |x|^{-1}\dot{\Gamma}(|x|)\, dx \\
&= 2\pi \left[\Gamma(b) - \Gamma(r)\right].
\end{aligned} \tag{2.7.20}$$

In the above $\mathbf{X}_r^b = \{x \in \mathbf{X} : r < |x| < b\}$. Now since,

$$2\pi\left[\Gamma(b) - \Gamma(a)\right] = \int_{\mathbf{X}} |x|^{-1}\dot{\Gamma}(|x|)dx = \int_{\mathbf{X}} |u|^{-1}\dot{\Gamma}(|u|)dx, \tag{2.7.21}$$

by Proposition 2.7.1 we obtain upon using (2.7.20) in (2.7.21) that,

$$\int_{\mathbf{X}_r} |u|^{-1}\dot{\Gamma}(|u|)dx \geq \int_{\mathbf{X}_r} |x|^{-1}\dot{\Gamma}(|x|)\,dx = 2\pi\left[\Gamma(r)-\Gamma(a)\right]. \tag{2.7.22}$$

Therefore,

$$\int_0^{2\pi} \Gamma(|u|)\frac{u\times u_\theta}{|u|^2}\,d\theta\,\bigg|_a^r \geq 2\pi\left[\Gamma(r)-\Gamma(a)\right], \tag{2.7.23}$$

but from the identity boundary conditions on u we know that,

$$\int_0^{2\pi} \Gamma(|u|)\frac{u\times u_\theta}{|u|^2}\,d\theta\,\bigg|_a = 2\pi\Gamma(a), \tag{2.7.24}$$

and therefore,

$$\int_0^{2\pi} \Gamma(|u|)\frac{u\times u_\theta}{|u|^2}\,d\theta\,\bigg|_r \geq 2\pi\Gamma(r), \qquad a.e.\ r\in[a,b]. \tag{2.7.25}$$

The result then follows from an application of Jensen's inequality. □

With the aid of these bounds we can now move on to the main goal of the section, namely, formulating and proving minimality for twist maps in homotopy classes of $\mathcal{A}(\mathbf{X})$ for a larger class of energies than those considered earlier.

Theorem 2.7.5. *Let $\mathbf{X}=\mathbf{X}[a,b]\subset\mathbb{R}^2$ and let $\mathbb{H}=\mathbb{H}[u;\mathbf{X}]$ denote the energy functional,*

$$\mathbb{H}[u;\mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} \Phi(|u|)\left[|\nabla|u||^2 + \frac{(u\times u_\theta)^2}{r^2|u|^2}\right] + \frac{(u\times u_r)^2}{|u|^4}\,dx, \tag{2.7.26}$$

where u lies in $\mathcal{A}(\mathbf{X})$ and $\Phi(t)=t^{-2}\Gamma(t)^2$ with $\Gamma(t)\in\mathbf{C}^2[a,b]$ such that $\dot{\Gamma}(t)/t$ is monotone increasing. Then for any $u\in\mathscr{A}_k$ $(k\in\mathbb{Z})$ there exists a twist map $\bar{u}=\bar{u}_k=\mathrm{Q}[g]x$ defined by the same symmetrisation as in (2.5.1) *such that,*

$$\mathbb{H}[\bar{u};\mathbf{X}] \leq \mathbb{H}[u;\mathbf{X}] \tag{2.7.27}$$

whilst $\bar{u}\in\mathscr{A}_k$. (Note that taking $\Phi(t)=1/t^2$, i.e., $\Gamma(t)=1$, gives $\mathbb{H}=\mathbb{F}$.)

Proof. As the first step in the proof we wish to prove the inequality

$$\int_{\mathbf{X}} \Phi(|x|)\,dx = \int_{\mathbf{X}} \Phi(|\bar{u}(x)|)|\nabla|\bar{u}(x)||^2\,dx \leq \int_{\mathbf{X}} \Phi(|u|)|\nabla|u(x)||^2\,dx. \tag{2.7.28}$$

In order to do this we again need to apply the isoperimetric inequality, the coarea formula for Sobolev functions as in the proof of Proposition 2.5.1 and then the integral identity (2.7.1). Thus we proceed by writing

$$\begin{aligned}\int_{\mathbf{X}} \Phi(|u|)|\nabla|u(x)||\,dx &= \int_a^b \Phi(t)\mathcal{H}^1(|u|=t)\,dt\\ &\geq \int_a^b \Phi(t)\mathcal{H}^1(|x|=t)\,dt\\ &= \int_{\mathbf{X}} \Phi(|x|)\,dx.\end{aligned} \tag{2.7.29}$$

Therefore it follows from basic considerations that

$$\begin{aligned}\left(\int_{\mathbf{X}} \Phi(|x|)dx\right)^2 &\le \left(\int_{\mathbf{X}} \Phi(|u|)|\nabla|u(x)||\,dx\right)^2\\ &\le \int_{\mathbf{X}} \Phi(|u|)|\nabla|u(x)||^2 dx \int_{\mathbf{X}} \Phi(|u|)\,dx\\ &= \int_{\mathbf{X}} \Phi(|u|)|\nabla|u(x)||^2 dx \int_{\mathbf{X}} \Phi(|x|)\,dx, \end{aligned}\tag{2.7.30}$$

and so rearranging gives the desired inequality (2.7.28), namely,

$$\int_{\mathbf{X}} \Phi(|x|)dx \le \int_{\mathbf{X}} \Phi(|u|)|\nabla|u(x)||^2\,dx. \tag{2.7.31}$$

Next we proceed by writing

$$\int_{\mathbf{X}} \Phi(|u|)\frac{(u\times u_\theta)^2}{r^2|u|^2}\,dx = \int_a^b \frac{1}{r}\int_0^{2\pi} \Phi(|u|)\frac{(u\times u_\theta)^2}{|u|^2}\,d\theta dr. \tag{2.7.32}$$

As $\Phi(t) = t^{-2}\Gamma(t)^2$ it follows upon noting Proposition 2.7.4 that we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \Phi(|u|)\frac{(u\times u_\theta)^2}{|u|^2}d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \Gamma(|u|)^2\frac{(u\times u_\theta)^2}{|u|^4}d\theta\\ &\ge \Gamma(r)^2 = \Phi(r)r^2, \quad a.e.\ r\in[a,b].\end{aligned}$$

Hence by combining the above it follows that

$$\int_{\mathbf{X}} \Phi(|u|)\frac{(u\times u_\theta)^2}{r^2|u|^2}\,dx \ge 2\pi\int_a^b \Phi(r)r\,dr = \int_{\mathbf{X}} \Phi(|\bar u|)\frac{(\bar u\times \bar u_\theta)^2}{r^2|\bar u|^2}\,dx. \tag{2.7.33}$$

Therefore with the above at our disposal all that remains is to use the inequality

$$\int_{\mathbf{X}} \frac{(u\times u_r)^2}{|u|^4}\,dx \ge \int_{\mathbf{X}} \frac{(\bar u\times \bar u_r)^2}{|\bar u|^4}\,dx, \tag{2.7.34}$$

whose proof proceeds similar to that of Proposition 2.5.1 by using the same angle of rotation function (2.5.10) in defining $\bar u$. This therefore completes the proof. □

2.8 Measure preserving self-maps and twists on solid tori

In this section we propose and study extensions of twist maps to a larger class of domains. Recalling that an n-dimensions annulus takes the form $\mathbf{X} = [a,b]\times\mathbb{S}^{n-1}$ the natural extension here would be domains of the product type $\mathbf{X} = \mathbb{B}^m\times\mathbb{S}^{n-1}$ (with $m\ge 1$, $n\ge 2$) embedded in $\mathbb{R}^{m+n}$. Twist maps in turn will be suitable measure preserving self-maps of $\overline{\mathbf{X}}$ that agree with the identity map ϕ on the boundary $\partial\mathbf{X}$ (*see* [74, 76]). To keep the discussion tractable we confine ourselves here to the case $m+n=3$. We proceed by first considering the solid torus $\mathbf{T}\cong\mathbb{B}^2\times\mathbb{S}^1$ embedded in $\mathbb{R}^3$ as (see Fig. 1):

$$\mathbf{T} = \left\{x = (x_1,x_2,x_3) : (\sqrt{x_1^2+x_2^2}-\rho)^2 + x_3^2 = r^2, \quad 0\le r<1\right\}. \tag{2.8.1}$$

Here $\mathbf{T} = \mathbf{T}_\rho$ and the fixed parameter ρ is chosen $\rho > 1$ to avoid self-intersection. Now let us set $\mu = \sqrt{x_1^2 + x_2^2} - \rho$. Then $\mathbf{T}$ above can be represented as

$$0 \le \mu^2 + x_3^2 = r^2 < 1. \tag{2.8.2}$$

From now on (μ, x_3) is the preferred choice of co-ordinates for $\mathbb{B} = \mathbb{B}_1^2$ where $\mathbb{B} = \{(\mu, x_3) \in \mathbb{R}^2 : \mu^2 + x_3^2 < 1\}$ is unit ball in the (μ, x_3) plane. In polar co-ordinates we have $\mu = r\cos\theta$, $x_3 = r\sin\theta$ and upon noting $\mu = \sqrt{x_1^2 + x_2^2} - \rho$ we have (x_1, x_2) as the co-ordinates of a sphere of radius $\rho + r\cos\theta$:

$$x_1 = (\rho + r\cos\theta)\cos\phi, \quad x_2 = (\rho + r\cos\theta)\sin\phi. \tag{2.8.3}$$

For our purposes in this section we shall write the above co-ordinate system in the following way,

$$\begin{cases} x_1 = (\mu + \rho)\cos\phi \\ x_2 = (\mu + \rho)\sin\phi \\ x_3 = x_3. \end{cases} \tag{2.8.4}$$

Now with the above notation in place we can define the desired twist maps on the solid torus $\mathbf{T}$ as, (Note that the aim here is to seek non-trivial extremising twist maps for the energy functional $\mathbb{F}$ over the admissible class of maps $\mathcal{A}(\mathbf{T})$.)

$$u(x) = \mathrm{Q}(\mu, x_3)x, \qquad x \in \mathbf{T}, \tag{2.8.5}$$

where the rotation matrix Q in $\mathbf{SO}(3)$ takes the explicit form

$$\mathrm{Q}(\mu, x_3) = \begin{bmatrix} \cos g(\mu, x_3) & -\sin g(\mu, x_3) & 0 \\ \sin g(\mu, x_3) & \cos g(\mu, x_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.8.6}$$

Here the function $g = g(\mu, x_3)$ defines the angle of rotation as in the case for the annulus, however, in this case g depends on the two variables (μ, x_3) and not just one $r = |x|$ as is the case for the annulus. There are two main reasons for this choice of representation of a twist map for $\mathbf{T}$, which we describe below.

- Firstly in order to be consistent with twist maps for the annulus we require that the rotation matrix is an isometry of the boundary

$$\partial\mathbf{T} = \left\{ x = (x_1, x_2, x_3) : (\sqrt{x_1^2 + x_2^2} - \rho)^2 + x_3^2 = 1 \right\}, \tag{2.8.7}$$

with respect to the metric induced from its embedding in $\mathbb{R}^3$. (This is similar to what was done earlier in the case of an annulus). Then with this in mind the isometries of $\partial\mathbf{T}$ are $\mathbf{SO}(3)$ matrices of the form,

$$\mathrm{Q} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.8.8}$$

- Secondly we allow the angle of rotation function g here to depend on the two variables (μ, x_3) instead of one to incorporate all the "*ball*" variables in the product structure on $\mathbf{T}$. Note that (μ, x_3) collapses into $r = |x|$ in the annulus case as here one deals with an interval (a one dimensional ball).

Now in preparation for the upcoming calculations let us denote $y = [x_1, x_2, 0]^t$, $\vartheta = y/|y|$ and $g_\mu = \partial g/\partial\mu$. Then it is easily seen that

$$\begin{cases} \nabla u = \mathrm{Q} + \dot{\mathrm{Q}}x \otimes \nabla g, \\ (\nabla u)(\nabla u)^t u = \mathrm{Q}x + \langle \nabla g, x\rangle \dot{\mathrm{Q}}x, \\ |\nabla u|^2 = 3 + (\mu+\rho)^2 |\nabla g|^2, \\ |u|^2 = (\mu+\rho)^2 + x_3^2, \\ \det(\nabla u) = \det\left(\mathrm{Q} + \dot{\mathrm{Q}}x \otimes \nabla g\right) \\ \qquad\qquad = 1 + \langle \mathrm{Q}^t\dot{\mathrm{Q}}x, \nabla g\rangle = 1. \end{cases} \tag{2.8.9}$$

Note that the last equality results from the fact that the product $\mathrm{Q}^t\dot{\mathrm{Q}}$ is skew-symmetric and $\nabla g = [(\mu+\rho)^{-1}x_1 g_\mu, (\mu+\rho)^{-1}x_2 g_\mu, g_{x_3}]^t$. Therefore using the above it is seen that the energy for a twist maps is given by

$$\mathbb{F}[u; \mathbf{T}] = \frac{1}{2}\int_{\mathbf{T}} \frac{|\nabla u|^2}{|u|^2}\,dx = \pi \int_{\mathbb{B}} \frac{3 + (\mu+\rho)^2|\nabla g|^2}{(\mu+\rho)^2 + x_3^2}(\mu+\rho)\,d\mu dx_3. \tag{2.8.10}$$

Here we are using the fact that a change in the co-ordinates $(r, \theta, \phi) \to (\mu, x_3, \phi)$ results in a Jacobian factor of $\mu + \rho$ in the integral, i.e.,

$$\int_0^{2\pi}\int_0^{2\pi}\int_{-1}^{1} r(\rho + r\cos\theta)\,drd\theta d\phi = \int_0^{2\pi}\int_{\mathbb{B}} (\mu+\rho)\,d\mu dx_3 d\phi.$$

Hence (2.8.10) becomes,

$$\mathbb{F}[u; \mathbf{T}] = \pi \int_{\mathbb{B}} \frac{(\mu+\rho)^3(g_\mu^2 + g_{x_3}^2)}{(\mu+\rho)^2 + x_3^2}\,d\mu dx_3 + \frac{3}{2}\int_{\mathbf{T}} |x|^{-2}dx, \tag{2.8.11}$$

where here the additional absolute constant does not affect the variational structure of $\mathbb{F}$. Now to derive the Euler-Lagrange equation associated to the energy integral on the right it suffices to take variations of $g = g(\mu, x_3)$ by some $\varphi \in \mathbf{C}_c^\infty(\mathbb{B})$. This calculation leads to the following divergence form equation:

$$\begin{aligned}
\frac{\partial}{\partial \mu}\left(\frac{(\mu+\rho)^3 g_\mu}{(\mu+\rho)^2+x_3^2}\right) + \frac{\partial}{\partial x_3}\left(\frac{(\mu+\rho)^3 g_{x_3}}{(\mu+\rho)^2+x_3^2}\right) &= 0,\\
\implies \operatorname{div}\frac{(\mu+\rho)^3 \nabla g}{(\mu+\rho)^2+x_3^2} &= 0. \qquad (2.8.12)
\end{aligned}$$

Evidently the identity boundary condition on u translates into $g(z) = 2k\pi$ for some fixed $k \in \mathbb{Z}$ and all $z = (\mu, x_3) \in \partial\mathbb{B}$. Now suppose g solves (2.8.12). Then by an application of the divergence theorem it is seen that the only solution to this boundary value problem is the trivial one, namely, $g(\mu, x_3) = 2\pi k$ for all $(\mu, x_3) \in \mathbb{B}$. Indeed

$$\begin{aligned}
0 &= \int_{\mathbb{B}} \operatorname{div}\left(\frac{(\mu+\rho)^3 \nabla g}{(\mu+\rho)^2+x_3^2}\right) d\mu dx_3\\
&= \int_0^{2\pi} \frac{(\cos\theta+\rho)^3}{(\cos\theta+\rho)^2+x_3^2}\frac{\partial g}{\partial r}(1,\theta)\, d\theta. \qquad (2.8.13)
\end{aligned}$$

Now again as g solves (2.8.12) an application of the divergence theorem also gives

$$\begin{aligned}
\int_{\mathbb{B}} \frac{(\mu+\rho)^3(g_\mu^2+g_{x_3}^2)}{(\mu+\rho)^2+x_3^2}\, d\mu dx_3 &= \int_{\mathbb{B}} \left[\frac{(\mu+\rho)^3(g_\mu^2+g_{x_3}^2)}{(\mu+\rho)^2+x_3^2} + g\operatorname{div}\frac{(\mu+\rho)^3\nabla g}{(\mu+\rho)^2+x_3^2}\right] d\mu dx_3\\
&= \int_0^{2\pi} g(1,\theta)\frac{(\cos\theta+\rho)^3}{(\cos\theta+\rho)^2+x_3^2}\frac{\partial g}{\partial r}(1,\theta)\, d\theta\\
&= 2\pi k \int_0^{2\pi} \frac{(\cos\theta+\rho)^3}{(\cos\theta+\rho)^2+x_3^2}\frac{\partial g}{\partial r}(1,\theta)\, d\theta = 0.
\end{aligned}$$

Note that in obtaining the last identity we have used (2.8.13) combined with the boundary condition satisfied by g, namely, $g(1, \theta) = 2\pi k$ for $0 \le \theta \le 2\pi$. Hence

$$\int_{\mathbb{B}} \frac{(\mu+\rho)^3(g_\mu^2+g_{x_3}^2)}{(\mu+\rho)^2+x_3^2}\, d\mu dx_3 = 0. \qquad (2.8.14)$$

Now since by assumption $\rho > 1$ we have $(\rho + \mu) > 0$ as $|\mu| = |r\cos\theta| \le r < 1$ and so $\mu > -1$. Hence (2.8.14) gives $|\nabla g|^2 = 0$ and thus $g(\mu, x_3) = 2\pi k$; again by invoking the boundary condition on g. It therefore follows that here we have no non-trivial solutions. Interestingly note that this conclusion stems from one crucial difference between the annulus $\mathbf{X}$ and the solid torus $\mathbf{T}$ in that $\mathbf{X}$ has two boundary components whilst $\mathbf{T}$ only has one. It was precisely this difference that turned crucial in the application of the divergence theorem.

Theorem 2.8.1. *There are no non-trivial twist solutions* (2.8.5) *to the Euler-Lagrange equations associated with the energy functional* $\mathbb{F}$ *on a solid torus* $\mathbf{T}$.

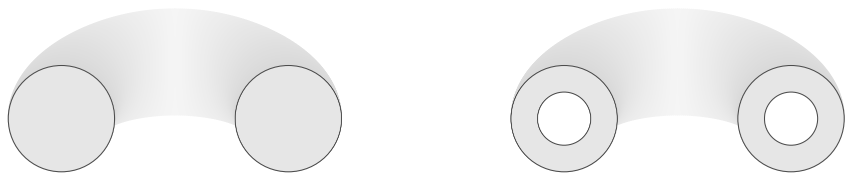


Figure 2.1: The solid torus (left) has a connected space of self-maps $\mathscr{C}(\mathbf{T})$, whereas for a "*thickened*" torus (right) the space of self-maps $\mathscr{C}(\mathbb{T})$ has infinitely many, indeed, $\mathbb{Z} \oplus \mathbb{Z}$ components. (See [74] and [76] for more.)

2.9 Twist maps on tori with disconnected double component boundary

In contrast to what was seen above let us next move on to considering a "*thickened*" torus, that is, the domain obtained topologically by taking the product of a two-dimensional torus and an interval. Note that here the boundary of the resulting domain consists of two disjoint copies of the initial torus and is in particular not connected. Now for definiteness and to fix notation let us set $\mathbb{T} = \mathbb{T}_\rho$ to be (see Fig. 1)

$$\mathbb{T} = \left\{ x = (x_1, x_2, x_3) : (\sqrt{x_1^2 + x_2^2} - \rho)^2 + x_3^2 = r^2, \quad a < r < 1 \right\}. \tag{2.9.1}$$

Here $0 < a < 1 < \rho$ are fixed and the aim is to seek non-trivial extremising twist maps for the energy functional $\mathbb{F}$ over the admissible class of maps $\mathcal{A}(\mathbb{T})$. Using the same co-ordinate system as in the earlier case we see that the (μ, x_3) are the co-ordinates of the two dimensional annulus centred at the origin. Additionally for similar reasons to that discussed earlier we define twist maps on $\mathbb{T}$ as $u(x) = \mathrm{Q}(\mu, x_3)x$ where the rotation matrix $\mathrm{Q} = \mathrm{Q}(\mu, x_3)$ in $\mathbf{SO}(3)$ is as (2.8.6).

A straightforward calculation shows that the energy of a twist maps is given by the integral

$$\begin{aligned} \mathbb{F}[u; \mathbb{T}] &= \frac{1}{2} \int_{\mathbb{T}} \frac{|\nabla u|^2}{|u|^2} \, dx = \pi \int_{\mathbb{B}_1 \setminus \overline{\mathbb{B}}_a} \frac{3 + (\mu + \rho)^2 |\nabla g|^2}{(\mu + \rho)^2 + x_3^2} (\mu + \rho) \, d\mu dx_3 \\ &= \pi \int_{\mathbb{B}_1 \setminus \overline{\mathbb{B}}_a} \frac{(\mu + \rho)^3 (g_\mu^2 + g_{x_3}^2)}{(\mu + \rho)^2 + x_3^2} \, d\mu dx_3 + \frac{3}{2} \int_{\mathbb{T}} |x|^{-2} dx. \end{aligned} \tag{2.9.2}$$

Similar to what was described earlier in obtaining the second equality we have used the integral identity

$$\int_0^{2\pi} \int_0^{2\pi} \int_a^1 r(\rho + r\cos(\theta)) \, dr d\theta d\phi = \int_0^{2\pi} \int_{\mathbb{B}_1 \setminus \overline{\mathbb{B}}_a} (\mu + \rho) \, d\mu dx_3 d\phi.$$

The Euler-Lagrange equation can be obtained in the standard way by taking variations $\varphi \in \mathbf{C}_c^\infty(\mathbb{B}_1\backslash\overline{\mathbb{B}}_a)$ where $\mathbb{B}_1\backslash\overline{\mathbb{B}}_a = \{(\mu, x_3) \in \mathbb{R}^2 : a^2 < \mu^2 + x_3^2 < 1\}$. This calculation again leads to the Euler-Lagrange equation given by (2.8.12) where we assume without loss of generality that the boundary condition on the rotation angle function g is set to $g(\mu, x_3) = 0$ for $(\mu, x_3) \in \partial\mathbb{B}_a$ and $g(\mu, x_3) = 2\pi k$ for $(\mu, x_3) \in \partial\mathbb{B}_1$ with $k \in \mathbb{Z}$. Therefore solutions to (2.8.12) satisfy,

$$0 = \int_{\mathbb{B}_1\backslash\overline{\mathbb{B}}_a} \operatorname{div}\frac{(\mu+\rho)^3\nabla g}{(\mu+\rho)^2+x_3^2}\,dx = \int_{\partial\mathbb{B}_1}\frac{(\mu+\rho)^3\nabla g\cdot n}{(\mu+\rho)^2+x_3^2} - \int_{\partial\mathbb{B}_a}\frac{(\mu+\rho)^3\nabla g\cdot n}{(\mu+\rho)^2+x_3^2}. \tag{2.9.3}$$

Subsequently

$$\begin{aligned}\frac{\mathbb{F}[u;\mathbb{T}] - 3/2\int_{\mathbb{T}}|x|^{-2}dx}{\pi} &= \int_{\mathbb{B}_1\backslash\overline{\mathbb{B}}_a}\left[\frac{(\mu+\rho)^3|\nabla g|^2}{(\mu+\rho)^2+x_3^2} + g\operatorname{div}\left(\frac{(\mu+\rho)^3\nabla g}{(\mu+\rho)^2+x_3^2}\right)\right]dx\\ &= \int_{\partial\mathbb{B}_1}\frac{g(\mu+\rho)^3\nabla g\cdot n}{(\mu+\rho)^2+x_3^2}\,d\mathcal{H}^1 - \int_{\partial\mathbb{B}_a}\frac{g(\mu+\rho)^3\nabla g\cdot n}{(\mu+\rho)^2+x_3^2}\,d\mathcal{H}^1.\end{aligned}$$

Hence taking into account the boundary conditions, e.g. $g = 0$ on $\partial\mathbb{B}_a$ and $g = 2\pi k$ on $\partial\mathbb{B}_1$, we gain that if g is a solution of (2.8.12) then,

$$\frac{\mathbb{F}[u;\mathbb{T}] - 3/2\int_{\mathbb{T}}|x|^{-2}dx}{\pi} = 2\pi k\int_{\partial\mathbb{B}_1}\frac{(\mu+\rho)^3\nabla g\cdot n}{(\mu+\rho)^2+x_3^2}\,d\mathcal{H}^1, \qquad k \in \mathbb{Z}. \tag{2.9.4}$$

Evidently (2.8.12) with the stated boundary conditions has a unique solution. Indeed if $\overline{g}, \underline{g}$ are two solutions to (2.8.12) with $g = 0$ on $\partial\mathbb{B}_a$ and $g = 2\pi k$ on $\partial\mathbb{B}_1$ then $g = \overline{g} - \underline{g}$ solves (2.8.12) with $g = 0$ on $\partial[\mathbb{B}_1\backslash\overline{\mathbb{B}}_a]$. Then by (2.9.4)

$$\int_{\mathbb{B}_1\backslash\overline{\mathbb{B}}_a}\frac{(\mu+\rho)^3|\nabla g|^2}{(\mu+\rho)^2+x_3^2}\,dx = 0. \tag{2.9.5}$$

However in view of $\rho > 1$ this gives $|\nabla g|^2 \equiv 0$ and so invoking the boundary conditions $g = 0$, i.e., $\overline{g} = \underline{g}$. As existence follows from standard arguments it follows that (2.8.12) has a unique smooth solution $g = g(\mu, x_3; k)$ for each $k \in \mathbb{Z}$.

2.10 Euler-Lagrange analysis and twists as classical solutions

The goal of this section is to examine the solution $g = g(\mu, x_3; k)$ to (2.8.12) with the prescribed boundary conditions in relation to the Euler-Lagrange system (2.4.6) associated with $\mathbb{F}$ on $\mathcal{A}(\mathbb{T})$. To this end recall that the system takes the form

$$\frac{(\nabla u)^t}{|u|^2}\left[\Delta u + \frac{|\nabla u|^2}{|u|^2}u - \frac{2}{|u|^2}\nabla u(\nabla u)^t u\right] = \nabla p. \tag{2.10.1}$$

For the ease of notation from now on we shall set $\xi = \mu + \rho$. Hence using the identities (2.8.9) we have

$$\begin{cases} (\nabla u)(\nabla u)^t u = \mathrm{Q}x + \langle \nabla g, x \rangle \dot{\mathrm{Q}}x, \\ |\nabla u|^2 |u|^{-2} = (3 + \xi^2 |\nabla g|^2)|x|^{-2}, \\ \Delta u = 2\xi^{-1} g_\xi \dot{\mathrm{Q}}x + \Delta g \dot{\mathrm{Q}}x + |\nabla g|^2 \ddot{\mathrm{Q}}x. \end{cases} \tag{2.10.2}$$

Therefore from (2.10.2) and a basic calculation we obtain

$$\Delta u + \frac{|\nabla u|^2}{|u|^2} u - \frac{2}{|u|^2} \nabla u (\nabla u)^t u = \left(\frac{2g_\xi}{\xi} + \Delta g - \frac{2\langle \nabla g, x \rangle}{|x|^2} \right) \dot{\mathrm{Q}}x + |\nabla g|^2 \ddot{\mathrm{Q}}x + \frac{1 + \xi^2 |\nabla g|^2}{|x|^2} \mathrm{Q}x.$$

Now since we have $\Delta g = \Delta_{\xi,x} g + g_\xi / \xi$ where the $\Delta_{\xi,x}$ denotes the Laplacian with respect to the ξ and x_3 variables we can rewrite this as

$$\Delta u + \frac{|\nabla u|^2}{|u|^2} u - \frac{2}{|u|^2} \nabla u (\nabla u)^t u = \left(\frac{3g_\xi}{\xi} + \Delta_{\xi,x} g - \frac{2\langle \nabla g, x \rangle}{|x|^2} \right) \dot{\mathrm{Q}}x + |\nabla g|^2 \ddot{\mathrm{Q}}x + \frac{1 + \xi^2 |\nabla g|^2}{|x|^2} \mathrm{Q}x. \tag{2.10.3}$$

Now upon recalling that the desired twist solution satisfies (2.8.12) we have that

$$\operatorname{div} \left(\frac{\xi^3 \nabla g}{\xi^2 + x_3^2} \right) = \frac{\xi^3 \Delta_{\xi,x} g}{\xi^2 + x_3^2} + \left(\frac{3\xi^2}{\xi^2 + x_3^2} - \frac{2\xi^4}{(\xi^2 + x_3^2)^2} \right) g_\xi - \frac{2\xi^3 x_3 g_{x_3}}{(\xi^2 + x_3^2)^2} = 0.$$

Thus dividing both sides by $\xi^3/(\xi^2 + x_3^2)$ and taking the negative terms to one side gives

$$\Delta_{\xi,x} g + \frac{3g_\xi}{\xi} = 2 \left(\frac{\xi g_\xi + x_3 g_{x_3}}{|x|^2} \right) = 2 \frac{\langle \nabla_{\xi,x} g, z \rangle}{|x|^2},$$

where $z = (\xi, x_3)^t$ and $\nabla_{\xi,x}$ denotes the gradient with respect to the (ξ, x_3) variable. Now since $\langle \nabla g, x \rangle = \langle \nabla_{\xi,x} g, z \rangle$ we obtain,

$$\Delta_{\xi,x} g + \frac{3g_\xi}{\xi} = 2 \frac{\langle \nabla g, x \rangle}{|x|^2},$$

and so as a result

$$\Delta u + \frac{|\nabla u|^2}{|u|^2} u - \frac{2}{|u|^2} \nabla u (\nabla u)^t u = |\nabla g|^2 \ddot{\mathrm{Q}}x + \frac{1 + \xi^2 |\nabla g|^2}{|x|^2} \mathrm{Q}x. \tag{2.10.4}$$

Next referring to the definition of Q basic calculation gives $\dot{\mathrm{Q}} = J_1 \mathrm{Q}$ and $\ddot{\mathrm{Q}} = -J_2 \mathrm{Q}$, where

$$J_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.10.5}$$

Hence with the above notation the Euler-Lagrange associated with the twist u, satisfying (2.8.12), simplifies to

$$
\begin{aligned}
\frac{(\nabla u)^t}{|x|^4}(\mathbf{I}_n + \xi^2|\nabla g|^2\mathbf{I}_n - |x|^2|\nabla g|^2 J_2)Qx &= (\mathbf{I}_n + \xi^2|\nabla g|^2\mathbf{I}_n - |x|^2|\nabla g|^2 J_2)\frac{x}{|x|^4} \\
&= \frac{1}{|x|^4}\begin{bmatrix}(1 - x_3^2|\nabla g|^2)x_1 \\ (1 - x_3^2|\nabla g|^2)x_2 \\ (1 + \xi^2|\nabla g|^2)x_3\end{bmatrix} \\
&= \nabla\left(-\frac{1}{2|x|^2}\right) + \frac{|\nabla g|^2}{|x|^4}\begin{bmatrix}-x_3^2x_1 \\ -x_3^2x_2 \\ \xi^2x_3\end{bmatrix} = \nabla p.
\end{aligned}
\tag{2.10.6}
$$

Considering the last line in the above equation it is plain that for u to grant solution to the Euler-Lagrange equation it must be that

$$
-\frac{1}{2}\nabla|x|^{-2} + \frac{|\nabla g|^2}{|x|^4}(-x_3^2x_1, -x_3^2x_2, \xi^2x_3)^t = \nabla p,
\tag{2.10.7}
$$

or equivalently that the second term on the left is a gradient. But for this to be the case the latter term must necessarily be curl-free and so this leads to the system of equations

$$
0 = \frac{\partial}{\partial x_1}\left(-\frac{x_3^2|\nabla g|^2}{|x|^4}x_2\right) - \frac{\partial}{\partial x_2}\left(-\frac{x_3^2|\nabla g|^2}{|x|^4}x_1\right),
\tag{2.10.8}
$$

$$
0 = \frac{\partial}{\partial x_1}\left(\frac{\xi^2|\nabla g|^2}{|x|^4}x_3\right) - \frac{\partial}{\partial x_3}\left(-\frac{x_3^2|\nabla g|^2}{|x|^4}x_1\right),
\tag{2.10.9}
$$

$$
0 = \frac{\partial}{\partial x_2}\left(\frac{\xi^2|\nabla g|^2}{|x|^4}x_3\right) - \frac{\partial}{\partial x_3}\left(-\frac{x_3^2|\nabla g|^2}{|x|^4}x_2\right).
\tag{2.10.10}
$$

It can be easily verified that equation (2.10.8) is satisfied for any twist map since here we have

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\left(-\frac{x_3^2|\nabla g|^2}{|x|^4}x_2\right) - \frac{\partial}{\partial x_2}\left(-\frac{x_3^2|\nabla g|^2}{|x|^4}x_1\right) \\
&= x_3^2|\nabla g|^2\left[\frac{\partial}{\partial x_2}\frac{x_1}{|x|^4} - \frac{\partial}{\partial x_1}\frac{x_2}{|x|^4}\right] + \frac{x_3^2}{|x|^4}\left[x_1\frac{\partial|\nabla g|^2}{\partial x_2} - x_2\frac{\partial|\nabla g|^2}{\partial x_1}\right] \\
&= \frac{x_3^2}{|x|^4}\left[x_1\frac{x_2}{\xi}\frac{\partial|\nabla g|^2}{\partial\xi} - x_2\frac{x_1}{\xi}\frac{\partial|\nabla g|^2}{\partial\xi}\right] = 0.
\end{aligned}
$$

We point out that the last line results upon noting the relations

$$
\frac{\partial}{\partial x_1} = \frac{x_1}{\xi}\frac{\partial}{\partial\xi} - \frac{x_2}{\xi^2}\frac{\partial}{\partial\phi}, \qquad \frac{\partial}{\partial x_2} = \frac{x_2}{\xi}\frac{\partial}{\partial\xi} + \frac{x_1}{\xi^2}\frac{\partial}{\partial\phi}.
\tag{2.10.11}
$$

Using this we can again see that we can write (2.10.9) and (2.10.10) as a single equation in the following way,

$$\frac{\partial}{\partial x_1}\left(\frac{\xi^2|\nabla g|^2}{|x|^4}x_3\right)+\frac{\partial}{\partial x_3}\left(\frac{x_3^2|\nabla g|^2}{|x|^4}x_1\right)= \\ =\frac{x_1x_3}{\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2|\nabla g|^2}{|x|^4}\right)+x_1\frac{\partial}{\partial x_3}\left(\frac{x_3^2|\nabla g|^2}{|x|^4}\right), \tag{2.10.12}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\xi^2|\nabla g|^2}{|x|^4}x_3\right)-\frac{\partial}{\partial x_3}\left(-\frac{x_3^2|\nabla g|^2}{|x|^4}x_2\right)= \\ =\frac{x_2x_3}{\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2|\nabla g|^2}{|x|^4}\right)+x_2\frac{\partial}{\partial x_3}\left(\frac{x_3^2|\nabla g|^2}{|x|^4}\right). \tag{2.10.13}$$

Therefore it is apparent that (2.10.9) and (2.10.10) become

$$x_3\frac{\partial}{\partial\xi}\left(\frac{\xi^2|\nabla g|^2}{|x|^4}\right)+\xi\frac{\partial}{\partial x_3}\left(\frac{x_3^2|\nabla g|^2}{|x|^4}\right)=0. \tag{2.10.14}$$

Now since we have the identities

$$\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{|x|^4}\right)=\frac{2\xi(x_3^2-\xi^2)}{(\xi^2+x_3^2)^3}=-\frac{\xi}{x_3}\frac{\partial}{\partial x_3}\left(\frac{x_3^2}{|x|^4}\right), \tag{2.10.15}$$

we obtain that (2.10.14) simplifies further to

$$\frac{x_3\xi}{|x|^4}\left[\xi\frac{\partial|\nabla g|^2}{\partial\xi}+x_3\frac{\partial|\nabla g|^2}{\partial x_3}\right]=0 \implies \xi\frac{\partial|\nabla g|^2}{\partial\xi}+x_3\frac{\partial|\nabla g|^2}{\partial x_3}=0. \tag{2.10.16}$$

Hence for a solution g to (2.8.12) with the prescribed boundary conditions to furnish a solution to the Euler-Lagrange system (2.4.6) associated with $\mathbb{F}$ it is necessary for g to satisfy

$$\xi\frac{\partial|\nabla g|^2}{\partial\xi}+x_3\frac{\partial|\nabla g|^2}{\partial x_3}=0. \tag{2.10.17}$$

We now show that (2.10.17) is also sufficient. Indeed assuming (2.10.17) the desired conclusion will follow upon showing that (2.10.7) holds. Towards this end set f to be the function,

$$f(\xi,x_3)=-\int_0^\xi\frac{x_3^2|\nabla g|^2}{(\tau^2+x_3^2)^2}\tau\,d\tau. \tag{2.10.18}$$

Then one can easily verify that,

$$\frac{\partial f}{\partial x_1}=-\frac{x_3^2|\nabla g|^2}{|x|^4}x_1,\quad \frac{\partial f}{\partial x_2}=-\frac{x_3^2|\nabla g|^2}{|x|^4}x_2. \tag{2.10.19}$$

Furthermore using (2.10.14) it is plain that

$$\frac{\partial f}{\partial x_3}=-\int_0^\xi\frac{\partial}{\partial x_3}\frac{x_3^2|\nabla g|^2\tau}{(\tau^2+x_3^2)^2}\,d\tau \\ =\int_0^\xi\frac{\partial}{\partial\tau}\frac{\tau^2|\nabla g|^2x_3}{(\tau^2+x_3^2)^2}\,d\tau=\frac{\xi^2|\nabla g|^2}{|x|^4}x_3. \tag{2.10.20}$$

As a result $\nabla f=|\nabla g|^2|x|^{-4}(-x_3^2x_1,-x_3^2x_2,\xi^2x_3)^t$.

Theorem 2.10.1. *A twist map u with the corresponding angle of rotation function $g=g(\mu,x_3;k)$ satisfying* (2.8.12) *and $g=0$ on $\partial\mathbb{B}_a$, $g=2\pi k$ on $\partial\mathbb{B}_1$ (with $k\in\mathbb{Z}$) is a solution to the Euler-Lagrange system* (2.4.6) *associated with $\mathbb{F}$ on $\mathcal{A}(\mathbb{T})$ iff it satisfies* (2.10.17).

Chapter 3

Whirl Mappings on Generalised Annuli and Incompressible Symmetric Equilibria of the Dirichlet Energy

Abstract

In this paper we show a striking contrast in the symmetry properties of extremisers and equilibria of the total elastic energy associated with an incompressible annulus. Indeed for the nonlinear Euler-Lagrange system associated with the elastic energy:

$$\mathbb{EL}[u, \mathbf{X}] = \begin{cases} \Delta u = div\ (\mathfrak{p}(x) \mathrm{cof}\, \nabla u) & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u \equiv x & \text{on } \partial \mathbf{X}. \end{cases}$$

where $\mathbf{X} = \{x \in \mathbb{R}^N : a < |x| < b\}$ we prove that, despite the inherent rotational symmetry, when $N = 3$ the problem possesses no non-trivial symmetric equilibria whereas in sharp contrast, when $N = 2$ the problem possesses an infinite family of such symmetric equilibria. We extend and prove the counterparts of these results in higher dimensions $N \geq 4$ and discuss a number of closely related issues.

Contents

3.1 Introduction

A problem of major interest and significance in nonlinear elasticity is the understanding of qualitative features and symmetries of energy minimiser and equilibrium states under the so-called incompressibility constraint (*see*, e.g., [1], [23], [64] and [68]). Motivated by the above and the earlier works in [64], [74]-[77], we in this paper make a small contribution towards certain aspects of this problem by considering the following geometric setup. We take $\mathbf{X} = \mathbf{X}[a,b] = \{(x_1, ..., x_N) : a < |x| < b\}$ with $0 < a < b < \infty$ fixed, to be an open annulus in $\mathbb{R}^N$ ($N \geq 2$) and consider the elastic energy

$$\mathbb{E}[u; \mathbf{X}] = \int_{\mathbf{X}} W(\nabla u)\, dx. \tag{3.1.1}$$

Here the stored energy density function $W : \mathbb{R}^{N\times N} \to \mathbb{R} \cup \{\infty\}$ is assumed isotropic, frame indifferent and polyconvex (*see* below) while the deformation u is restricted to lie in the class of incompressible Sobolev mappings

$$\mathcal{A}(\mathbf{X}) = \Big\{ u \in W^{1,2}(\mathbf{X}, \mathbb{R}^N) : \det \nabla u = 1 \ \textit{a.e.} \text{ in } \mathbf{X} \text{ and } u \equiv x \ \text{ on } \partial\mathbf{X} \Big\}, \tag{3.1.2}$$

where the last condition in (3.1.2) means that u agrees with the identity mapping on $\partial\mathbf{X}$ in the sense of traces. To prevent interpenetration of matter $W(\mathbf{F}) = \infty$ when $\det \mathbf{F} \leq 0$ and so in this regard it is only $\mathbb{R}^{N\times N}_+ = \{\mathbf{F} \in \mathbb{R}^{N\times N} : \det \mathbf{F} > 0\}$ that is of interest here. Recall that $W = W(\mathbf{F})$ is said to be polyconvex *iff* it is a convex function of all minors of $\mathbf{F}$, that is, there exists a convex function $\varphi : \mathbb{R}^{\tau(N)} \to \mathbb{R} \cup \{\infty\}$ such that, (note that

$\mathrm{adj}_s\mathbf{F}$ stands for the matrix of all $s \times s$ minors of $\mathbf{F}$ and $\tau(n) = \sum_{j=1}^{n} \binom{n}{s}^2$)

$$W(\mathbf{F}) = \varphi(\mathbf{F}, \mathrm{adj}_2\mathbf{F}, \cdots, \mathrm{adj}_N\mathbf{F}), \tag{3.1.3}$$

while W is isotropic and frame indifferent *iff* for all $\mathrm{Q} \in \mathbf{SO}(N)$ respectively

$$W(\mathrm{Q}\mathbf{F}) = W(\mathbf{F}), \qquad W(\mathbf{F}\mathrm{Q}) = W(\mathbf{F}), \tag{3.1.4}$$

that is, W is invariant under both the left and right action of $\mathbf{SO}(N)$ (for further discussion and representations *cf.*, e.g., [1, 6, 10, 23] or [24]). A particular example of an isotropic, frame invariant and polyconvex stored energy density function is give by

$$W(\mathbf{F}) = tr\{\mathbf{F}^t\mathbf{F}\} + h(\det \mathbf{F}) = \sum_{j=1}^{N} v_j^2 + h(\prod_{j=1}^{N} v_j), \tag{3.1.5}$$

where h is any convex function on the line, $v_1, ..., v_N$ are the singular values of $\mathbf{F}$, that is, the eigenvalues of $\sqrt{\mathbf{F}^t\mathbf{F}}$ and the second equality assumes $\det \mathbf{F} > 0$ (*see*, [1, 7, 9, 10, 11] and [23, 24] for more).

In virtue of the incompressibility constraint in (3.1.2) and subject to the differentiability of the stored energy density W in (3.1.1) it can be seen, upon invoking the Lagrange multipliers method, that the Euler-Lagrange system associated with the elastic energy $\mathbb{E}[\cdot; \mathbf{X}]$ over the class of admissible deformations $\mathcal{A}(\mathbf{X})$ takes the form

$$\mathbb{EL}[u, \mathbf{X}] = \begin{cases} div\, \mathfrak{S}[x, \nabla u(x)] = 0 & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u \equiv x & \text{on } \partial\mathbf{X}, \end{cases} \tag{3.1.6}$$

where the divergence operator here is understood to act row-wise while the stress tensor field $\mathfrak{S}$ is given by

$$\mathfrak{S}[x, \mathbf{F}] = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - \mathfrak{p}(x)\mathbf{F}^{-t} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - \mathfrak{p}(x)\mathrm{cof}\, \mathbf{F}. \tag{3.1.7}$$

Furthermore the function $\mathfrak{p}$ above is an *apriori* unknown Lagrange multiplier associated with the incompressibility constraint and is often referred to as the hydrostatic pressure.

In this paper we shall introduce and study a class of mappings called *whirl* mappings (or *whirls* for simplicity). These are continuous self-mappings of the N-dimensional annulus $\overline{\mathbf{X}}$ (onto itself) agreeing with the identity on the boundary $\partial\mathbf{X}$ and having the specific representation

$$u : x \mapsto \mathrm{Q}(\rho_1, \ldots, \rho_n)\, x, \qquad x \in \overline{\mathbf{X}}. \tag{3.1.8}$$

Here Q is an $\mathbf{SO}(N)$-valued matrix field depending on the spatial variable x through the 2-plane radial variables $\varrho = (\rho_1, \cdots, \rho_n)$ where

$$\rho_j = \sqrt{x_{2j-1}^2 + x_{2j}^2}, \qquad 1 \le j \le n-1, \tag{3.1.9}$$

and

$$\rho_n = \begin{cases} \sqrt{x_{2n-1}^2 + x_{2n}^2} & \text{if } N = 2n, \\ x_{2n-1} & \text{if } N = 2n-1. \end{cases} \tag{3.1.10}$$

It is easily seen that the 2-plane radial variables $\varrho = (\rho_1, \ldots, \rho_n)$ lie in $U_N \subset \mathbb{R}^n$ where $U_N = \{\varrho \in \mathbb{R}^n_+ : a \le |\varrho| \le b\}$ when $N = 2n$ while $U_N = \{\varrho \in \mathbb{R}^{n-1}_+ \times \mathbb{R} : a \le |\varrho| \le b\}$ when $N = 2n-1$.

Next for the purpose of symmetry considerations described below we demand Q to take values on a fixed *maximal torus* $\mathbb{T} \subset \mathbf{SO}(N)$ that here we set to be the subgroup of all 2×2 block diagonal rotation matrices. As a result we can write

$$\mathrm{Q}(\varrho) = \mathrm{Q}(\rho_1, \ldots, \rho_n) = \mathrm{diag}(\mathbf{R}[g_1], \ldots, \mathbf{R}[g_n]), \qquad N = 2n, \tag{3.1.11}$$

and

$$\mathrm{Q}(\varrho) = \mathrm{Q}(\rho_1, \ldots, \rho_n) = \mathrm{diag}(\mathbf{R}[g_1], \ldots, \mathbf{R}[g_{n-1}], 1), \qquad N = 2n-1, \tag{3.1.12}$$

where each block $\mathbf{R} = \mathbf{R}[g] \in \mathbf{SO}(2)$ is determined by an *angle of rotation* or *whirl* function $g_j = g_j(\rho_1, \ldots, \rho_n)$. Now any such whirl mapping u is invariant under the action of the maximal torus $\mathbb{T}$ since firstly each ρ_j with $1 \le j \le n$ remains fixed under the action of $\mathbb{T}$ and secondly,

$$[\mathbf{P}u \circ \mathbf{P}^t](x) = \mathbf{P}u(\mathbf{P}^t x) = \mathbf{P}\mathrm{Q}(\rho_1, \ldots, \rho_n)\mathbf{P}^t x = \mathrm{Q}(\rho_1, \ldots, \rho_n)x = u(x),$$

for all $\mathbf{P} \in \mathbb{T}$ and $x \in \overline{\mathbf{X}}$. (Note that the second to last equality above follows from the commutativity of $\mathbb{T}$.) Therefore in this sense the whirl mappings as defined above are rotationally symmetry with respect to $\mathbb{T} \subset \mathbf{SO}(N)$.

Specialising to $N = 3$ the $\mathbf{SO}(3)$-valued matrix field Q defining u is expressed as $\mathrm{Q} = \mathrm{Q}[g]$ with $g = g(\rho_1, x_3)$ where $\rho_1 = \sqrt{x_1^2 + x_2^2}$ and $\rho_2 = x_3$ [*cf.* (3.2.4)]. Here the gradient of u is expressed as (with $\rho = \rho_1$, $\dot{\mathrm{Q}} = \partial_g \mathrm{Q}$)

$$\nabla u = \mathrm{Q}(\rho, x_3) + \dot{\mathrm{Q}}(\rho, x_3)x \otimes \nabla g. \tag{3.1.13}$$

Now for $x \in \mathbf{X}$ let $\mathbf{P}_1 = \mathbf{P}\mathbf{R}\mathrm{Q}^t$ and $\mathbf{P}_2 = \mathbf{R}^t\mathbf{P}^t$ where $\mathbf{R}, \mathbf{P} \in \mathbf{SO}(3)$ are chosen so that $\mathbf{R}(x_1, x_2, x_3)^t = (0, \sqrt{x_1^2 + x_2^2}, x_3)^t$ and $\mathbf{P}(x_1, x_2, x_3)^t = (x_1, 0, \sqrt{x_2^2 + x_3^2})^t$. Then noting

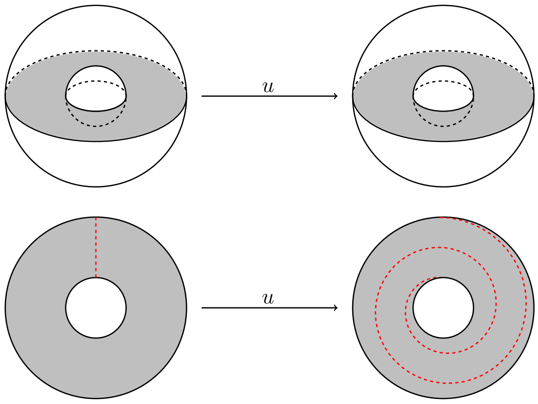


Figure 3.1: The figure here shows how a *whirl* mapping u acts on $\mathbf{X}$ when $N = 3$. In particular it shows how the *whirl* u wraps a radial line around the centre in the (x_1, x_2)-plane.

$\nabla g = [\rho^{-1} g_\rho \, x_1, \rho^{-1} g_\rho \, x_2, g_{x_3}]^t$ a straightforward calculation gives

$$\mathbf{P}_1 \nabla u \, \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & \rho|\nabla g| \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: \mathbb{B}[\rho \nabla g]. \tag{3.1.14}$$

Hence in view of the assumptions on W the elastic energy $\mathbb{E}$ when restricted to these whirl mappings in three dimensions is given by,

$$\mathbb{E}[u; \mathbf{X}] = \int_{\mathbf{X}} W(\nabla u) \, dx = \int_{\mathbf{X}} W(\mathbb{B}[\rho \nabla g]) \, dx = 2\pi \int_U f(\rho|\nabla g|) \rho \, d\rho dx_3 \tag{3.1.15}$$

Here $U = U_2$ as defined earlier is a half annulus (see also Section 3.2) and $f(t) = W(\mathbb{B}[t])$ is the restriction of the stored-energy function to the rank-one line through the identity given by $\mathbb{B}[t]$.

For the sake of this paper we shall confine to invariant stored energy densities of the form (3.1.5) that in view of the incompressibility constraint can be taken without any loss of generality to be $W(\mathbf{F}) = tr\{\mathbf{F}^t \mathbf{F}\}/2$ with the resulting elastic energy $\mathbb{E}$ in (3.1.1) being the Dirichlet energy. The main goal here is to highlight a stark difference in the existence of symmetric equilibria (critical points) to the elastic energy (3.1.1) or alternatively solutions to the nonlinear system (3.1.6) in the form of whirl mappings between the cases $N = 2$ and $N = 3$. Indeed in the latter case we show that there are no non-trivial critical points of the energy in the class of whirl mappings whereas in the former case they do exist and quite in abundance – in fact, here, we show that there are infinitely (countably) many equilibria in the form of whirl mappings. In the final sections of the paper we also establish the counterpart of these results in higher dimensions $N \geq 4$.

3.2 Existence and non-existence results in two and three dimensions

The aim of this section is to illustrate a sharp contrast in the nature of symmetric equilibria between the cases $N = 2$ and $N = 3$. In particular we will show that when $N = 3$ the only whirl solution to (3.2.1) below is the identity mapping whilst when $N = 2$ there are infinitely (countably) many such solutions. Towards this end let us recall that the Euler-Lagrange system associated with the Dirichlet energy ($W(\mathbf{F}) = tr\,(\mathbf{F}^t\mathbf{F})/2$) over $\mathcal{A}(\mathbf{X})$ (for any $N \geq 2$) is given by the nonlinear system

$$\begin{cases} div\ \mathfrak{S}[x, \nabla u(x)] = 0 & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u \equiv x & \text{on } \partial\mathbf{X}, \end{cases} \tag{3.2.1}$$

where the stress field $\mathfrak{S}$ in this case is given by $\mathfrak{S}[x, \mathbf{F}] = \mathbf{F} - \mathfrak{p}(x)\mathbf{F}^{-t}$. Therefore the divergence term on the left in the first line of the above system can be written as

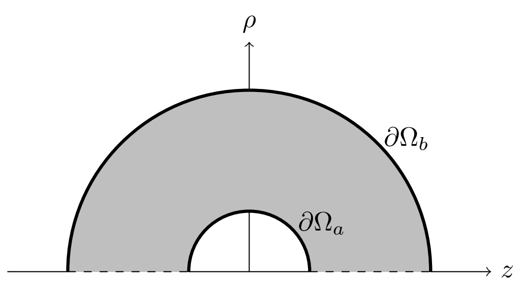
$$div\ \mathfrak{S}[x, \nabla u(x)] = \Delta u - \operatorname{div}(\mathfrak{p}(\nabla u)^{-t}) = \Delta u - (\nabla u)^{-t}\nabla\mathfrak{p}, \tag{3.2.2}$$

where the last identity subject to sufficient regularity of u and results from an application of the Piola identity. Hence the first equation in the system (3.2.1) reads as

$$div\ \mathfrak{S}[x, \nabla u(x)] = 0 \iff \nabla\mathfrak{p} = (\nabla u)^t\Delta u. \tag{3.2.3}$$

By a classical solution here we mean a pair $(u, \mathfrak{p})$ where u is admissible, that is, $u \in \mathcal{A}(\mathbf{X})$, u is regular, i.e., $u \in \mathbf{C}(\overline{\mathbf{X}}, \mathbb{R}^3) \cap \mathbf{C}^2(\mathbf{X}, \mathbb{R}^3)$, $\mathfrak{p} \in \mathbf{C}^1(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$ and (3.2.1) or the equivalent formulation of the first equation (3.2.3) hold. Now being prompted by symmetry considerations we wish to focus on solutions to (3.2.1) from the class of whirl mappings.

The case $N = 3$. Let $U = U[a, b] = \{(\rho, x_3) \in \mathbb{R}^2 : \rho \geq 0, a < \sqrt{\rho^2 + x_3^2} < b\}$, i.e., the *half* vertical open annulus whose closure upon a 2π rotation about the x_3-axis gives $\overline{\mathbf{X}}$.



Then by definition any whirl mapping u on $\mathbf{X}$ is represented as $u(x) = \mathrm{Q}(\rho, z)x$ where $x = (x_1, x_2, x_3)$, $\rho = \sqrt{x_1^2 + x_2^2}$ and for brevity $z = x_3$ where for each $(\rho, z) \in U$ the matrix field Q has the specific block diagonal form

$$\mathrm{Q} = \mathrm{Q}[g](\rho, z) = \mathrm{diag}(\mathbf{R}[g], 1) = \begin{bmatrix} \mathbf{R}[g] & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.2.4}$$

Here $\mathbf{R} = \mathbf{R}[g]$ is a planar rotation matrix with $g = g(\rho, z)$ representing the *rotation* angle (*whirl* function) belonging to the set

$$\mathbf{G}_k(U) := \Big\{ g \in W^{1,2}(U) : g \equiv 0 \text{ on } \partial U_a \text{ and } g \equiv 2\pi k \text{ on } \partial U_b \Big\}, \qquad k \in \mathbb{Z}, \tag{3.2.5}$$

while $\partial U_a = \{(\rho, x_3) : \sqrt{\rho^2 + x_3^2} = a\}$ and $\partial U_b = \{(\rho, x_3) : \sqrt{\rho^2 + x_3^2} = b\}$ and so the boundary conditions on $g \in \mathbf{G}_k$ ensure that $u \equiv x$ on $\partial\mathbf{X}$. Next a set of direct and straightforward calculations based on the representation of u and Q give

$$\begin{aligned}
\nabla u &= \left[\frac{\partial u_i}{\partial x_j} : 1 \le i, j \le 3 \right] \\
&= \left[\begin{matrix} \cos g - x_1(x_1 \sin g + x_2 \cos g) g_\rho/\rho & -\sin g - x_2(x_1 \sin g + x_2 \cos g) g_\rho/\rho \\ \sin g + x_1(x_1 \cos g - x_2 \sin g) g_\rho/\rho & +\cos g + x_2(x_1 \cos g - x_2 \sin g) g_\rho/\rho \\ 0 & 0 \end{matrix} \right. \\
&\qquad \left. \begin{matrix} -(x_1 \sin g + x_2 \cos g) g_z \\ +(x_1 \cos g - x_2 \sin g) g_z \\ 1 \end{matrix} \right], \\
&= \begin{bmatrix} \mathbf{R}[g](\rho, z) & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -(x_1 \sin g + x_2 \cos g) \\ x_1 \cos g - x_2 \sin g \\ 0 \end{bmatrix} \otimes \begin{bmatrix} x_1 g_\rho/\rho \\ x_2 g_\rho/\rho \\ g_z \end{bmatrix}.
\end{aligned} \tag{3.2.6}$$

The last equation in particular gives $\det \nabla u = 1$ in $\mathbf{X}$. (Note that for any invertible matrix $\mathcal{A}$ and vectors a, b: $\det(\mathcal{A} + a \otimes b) = \det \mathcal{A} \times \det(\mathrm{I} + \mathcal{A}^{-1} a \otimes b) = \det \mathcal{A} + \langle \mathcal{A}^{-1} a, b \rangle \det \mathcal{A}$ as a consequence of the determinant being quasiaffine.) Therefore it is evident that $u \in \mathcal{A}(\mathbf{X})$. Next a straightforward calculation gives

$$|\nabla u|^2 = \mathrm{tr}\{[\nabla u]^t [\nabla u]\} = 3 + \rho^2 (g_\rho^2 + g_z^2), \tag{3.2.7}$$

which upon integration and a change of variables gives

$$\begin{aligned}
2\mathbb{E}[u; \mathbf{X}] &= \int_{\mathbf{X}} |\nabla u|^2 \, dx = \int_{\mathbf{X}} [3 + \rho^2 (g_\rho^2 + g_z^2)] \, dx \\
&= 2\pi \int_U [3 + \rho^2 (g_\rho^2 + g_z^2)] \rho \, d\rho dz =: 3|\mathbf{X}| + 2\pi \mathbb{H}[g; U].
\end{aligned} \tag{3.2.8}$$

Now attempting to extremise $\mathbb{E}[u;\mathbf{X}]$ over $\mathcal{A}(\mathbf{X})$ and seeking solutions in the form of whirl mappings we proceed, in view of (3.2.8), by extermising the restrcited energy $\mathbb{H}[g;U]$ over the grand class $\mathbf{G}(U)$ of all angle of rotation functions:

$$\mathbb{H}[g;U] = \int_U \rho^3 |\nabla g|^2 \, d\rho dx_3, \qquad \mathbf{G}(U) = \bigcup_{k\in\mathbb{Z}} \mathbf{G}_k(U). \tag{3.2.9}$$

Evidently the Euler-Lagrange equation associated with $\mathbb{H}[\cdot;U]$ over $\mathbf{G}_k(U)$ is seen to take the divergence form

$$\begin{cases} \partial_\rho(\rho^3 g_\rho) + \partial_z(\rho^3 g_z) = 0 & \text{in } U, \\ g = 0 & \text{on } \partial U_a, \\ g = 2\pi k & \text{on } \partial U_b, \\ \rho^3 \partial_\nu g = 0 & \text{on } \partial U \backslash [\partial U_a \cup \partial U_b]. \end{cases} \tag{3.2.10}$$

Notice that the horizontal part of the boundary $\partial U\backslash[\partial U_a \cup \partial U_b] = \{(\rho, z) \in U : \rho = 0\}$ is left free accounting for the natural boundary condition in the last line. Any classical solution to (3.2.10) defines a corresponding whirl mapping u (as outlined earlier in the introduction) which is a possible candidate for a solution to (3.2.1). Furthermore a function g is referred to as a classical solution of (3.2.10) if $g \in \mathbf{C}^2(U)\cap\mathbf{C}(\overline{U})$ and (3.2.10) is satisfied.

The following proposition establishes the existence of a unique solution to the restricted Euler-Lagrange equation (3.2.10) for each fixed $k \in \mathbb{Z}$.

Proposition 3.2.1. *The restricted Euler-Lagrange equation* (3.2.10) *has a unique classical solution* $g = g(\rho, z; k)$ *in* $\mathbf{G}_k(U)$ *for each fixed* $k \in \mathbb{Z}$. *This solution is given explicitly by*

$$g(\rho, z; k) = \frac{2\pi k a^3 b^3}{b^3 - a^3} \left[\frac{1}{a^3} - \frac{1}{(\rho^2 + z^2)^{\frac{3}{2}}} \right], \qquad (\rho, z) \in \overline{U}. \tag{3.2.11}$$

Proof. First a straightforward calculation verifies that g satisfies the required boundary conditions in (3.2.10). Moreover for $\nabla g = (\partial_\rho g, \partial_z g)$ we have

$$\partial_\rho g = \frac{6\pi k a^3 b^3}{b^3 - a^3} \frac{\rho}{(\rho^2 + z^2)^{\frac{5}{2}}}, \quad \partial_z g = \frac{6\pi k a^3 b^3}{b^3 - a^3} \frac{z}{(\rho^2 + z^2)^{\frac{5}{2}}}, \tag{3.2.12}$$

and so clearly $g \in W^{1,2}(U)$. Next referring to the first equation in (3.2.10) it is plain that

$$\begin{aligned} div\left(\rho^3 \nabla g\right) =& \partial_\rho(\rho^3 \partial_\rho g) + \partial_z(\rho^3 \partial_z g) = \frac{6\pi k a^3 b^3}{b^3 - a^3} \times \\ & \times \left[\partial_\rho \{\rho^4 (\rho^2 + z^2)^{-5/2}\} + \partial_z \{\rho^3 z (\rho^2 + z^2)^{-5/2}\} \right] = 0 \end{aligned} \tag{3.2.13}$$

(The vector field $\rho^3(\rho^2 + z^2)^{-5/2}[\rho, z]^t$ is divergence free in U.) Thus g solves (3.2.10). The proof of uniqueness is standard: assume g^1, g^2 are classical solutions to (3.2.10) and set $g = g^1 - g^2$. Then g solves (3.2.10) with $g \equiv 0$ on $\partial U_a \cup \partial U_b$ and so an application of the divergence theorem along with the above gives

$$\int_U \rho^3 |\nabla g|^2 \, d\rho dz = \int_U \operatorname{div}\left(\rho^3 g \nabla g\right) \, d\rho dz = \int_{\partial U} \rho^3 g \nabla g \cdot \nu \, d\sigma = 0. \tag{3.2.14}$$

However as $\rho > 0$ in U and $|\nabla g|^2 \geq 0$ we must have $g \equiv c$ for some constant c that as a result of the zero boundary conditions gives $g \equiv 0$, i.e., $g^1 \equiv g^2$. For clarity we note that g begin a classical solution to (3.2.10) gives that the vector field $\mathbf{F} = \rho^3 g \nabla g \in \mathbf{C}^1(U, \mathbb{R}^3) \cap \mathbf{C}(\overline{U}, \mathbb{R}^3)$ with $\operatorname{div} \mathbf{F} \in L^1(U)$, which allows us to apply the Divergence Theorem above. $\square$

We are now in a position to prove that when $N = 3$ the whirl mapping u with the associated angle of rotation function g is not a solution to the system (3.2.1) except for $k = 0$.

Theorem 3.2.2. ($N = 3$) *Let* $\mathbf{X} = \mathbf{X}[a, b] \subset \mathbb{R}^3$ *be an open annulus and consider the elastic energy* (3.1.1) *with* $W(\mathbf{F}) = tr(\mathbf{F}^t \mathbf{F})/2$ *over the space of incompressible admissible mappings* $\mathcal{A}(\mathbf{X})$. *Then there are no non-trivial equilibria in the form of a whirl mapping.*

Proof. We begin by formulating the full Euler-Lagrange equation associated with the Dirichlet energy in terms of the whirl function g associated with the whirl mapping $u \in \mathcal{A}(\mathbf{X})$ and $x \in \mathbf{X} \backslash \{x : \rho = 0\}$. Towards this end we note that,

$$\nabla u(x) = \mathrm{Q}[g] + \dot{\mathrm{Q}}[g] x \otimes \nabla g, \tag{3.2.15}$$

$$\Delta u(x) = \Delta g \dot{\mathrm{Q}}[g] x + 2 \dot{\mathrm{Q}}[g] \nabla g + |\nabla g|^2 \ddot{\mathrm{Q}} x. \tag{3.2.16}$$

Furthermore an explicit calculations gives that,

$$\begin{aligned}(\nabla u)^t \Delta u = {} & 2 \mathrm{Q}^t \dot{\mathrm{Q}} \nabla g + \Delta g \mathrm{Q}^t \dot{\mathrm{Q}} x + |\nabla g|^2 \mathrm{Q}^t \ddot{\mathrm{Q}} x \\ & + 2 \langle \dot{\mathrm{Q}} x, \dot{\mathrm{Q}} \nabla g \rangle \nabla g + \Delta g |\dot{\mathrm{Q}} x|^2 \nabla g + |\nabla g|^2 \langle \dot{\mathrm{Q}} x, \ddot{\mathrm{Q}} x \rangle \nabla g. \end{aligned} \tag{3.2.17}$$

Also note that $\langle \dot{\mathrm{Q}} x, \ddot{\mathrm{Q}} x \rangle = 0$, $|\dot{\mathrm{Q}} x|^2 = \rho^2$ and $\langle \dot{\mathrm{Q}} x, \dot{\mathrm{Q}} \nabla g \rangle = \rho g_\rho$ and thus the above reduces to,

$$(\nabla u)^t \Delta u = 2 \mathrm{Q}^t \dot{\mathrm{Q}} \nabla g + \Delta g \mathrm{Q}^t \dot{\mathrm{Q}} x + |\nabla g|^2 \mathrm{Q}^t \ddot{\mathrm{Q}} x + \left(2 \rho g_\rho + \rho^2 \Delta g\right) \nabla g. \tag{3.2.18}$$

Moreover $\Delta g = \rho^{-1} g_\rho + g_{\rho\rho} + g_{zz}$ which by the reduced Euler-Lagrange system, given by (3.2.3), we know that,

$$\Delta g + 2 \frac{g_\rho}{\rho} = 0. \tag{3.2.19}$$

Therefore plugging (3.2.19) into (3.2.18) we obtain that,

$$(\nabla u)^t \Delta u = 2\mathrm{Q}^t\dot{\mathrm{Q}}\nabla g + \Delta g \mathrm{Q}^t\dot{\mathrm{Q}}x + |\nabla g|^2\mathrm{Q}^t\ddot{\mathrm{Q}}x, \tag{3.2.20}$$

or in components

$$\begin{aligned}(\nabla u)^t \Delta u &= \begin{bmatrix} -\rho^{-1}x_2(3g_\rho + \rho g_{\rho\rho} + \rho g_{zz}) + 2x_1 g_\rho^2 + \rho x_1 g_\rho(g_{\rho\rho} + g_{zz}) - x_1 g_z^2 \\ +\rho^{-1}x_1(3g_\rho + \rho g_{\rho\rho} + \rho g_{zz}) + 2x_2 g_\rho^2 + \rho x_2 g_\rho(g_{\rho\rho} + g_{zz}) - x_2 g_z^2 \\ \rho g_z(3g_\rho + \rho g_{\rho\rho} + \rho g_{zz}) \end{bmatrix} \\ &= \begin{bmatrix} -\rho^{-1}x_2(2g_\rho + \rho\Delta g) - x_1(g_\rho^2 + g_z^2) \\ \rho^{-1}x_1(2g_\rho + \rho\Delta g) - x_2(g_\rho^2 + g_z^2) \\ 0 \end{bmatrix}\end{aligned} \tag{3.2.21}$$

Hence after applying (3.2.19) to the above we obtain

$$\nabla\mathfrak{p} = (\nabla u)^t\Delta u = \begin{bmatrix} -x_1(g_\rho^2 + g_z^2) \\ -x_2(g_\rho^2 + g_z^2) \\ 0 \end{bmatrix}. \tag{3.2.22}$$

Here (3.2.22) is the formulation of the Euler-Lagrange system for the candidate whirl mapping u in terms of its associated whirl function g. Before we use the explicit form of the solutions g to the restricted Euler-Lagrange (3.2.10) we note that a necessary condition for the solvability of the above system for $\mathfrak{p}$ is that the vector field on the *right* must be *curl-free*. Thus,

$$\begin{aligned} \mathbf{curl}(\nabla\mathfrak{p}) = \nabla \times \big[(\nabla u)^t\Delta u\big] &= 0 \\ &\Longleftrightarrow \\ \frac{\partial}{\partial z}\Big[(g_\rho^2 + g_z^2)\Big] &= 0, \\ x_1\frac{\partial}{\partial x_2}\Big[(g_\rho^2 + g_z^2)\Big] - x_2\frac{\partial}{\partial x_1}\Big[(g_\rho^2 + g_z^2)\Big] &= 0. \end{aligned} \tag{3.2.23}$$

However since the whirl function g is explicitly given by (3.2.11), for each $k \in \mathbb{Z}$, a basic calculation gives

$$g_\rho^2 + g_z^2 = \left(\frac{6\pi k a^3 b^3}{b^3 - a^3}\right)^2 \frac{1}{(\rho^2 + z^2)^4}, \tag{3.2.24}$$

and so in particular when $k \neq 0$ we have

$$\frac{\partial}{\partial z}\Big[(g_\rho^2 + g_z^2)\Big] = -\left(\frac{6\pi k a^3 b^3}{b^3 - a^3}\right)^2 \frac{8z}{(\rho^2 + z^2)^5} \neq 0, \tag{3.2.25}$$

contradicting (3.2.23). Therefore as claimed there are no non-trivial whirl mappings as equilibria of the elastic energy $\mathbb{E}$ over $\mathcal{A}(\mathbf{X})$. $\square$

The case $N = 2$**.** Let us next contrast the non-existence result for (non-trivial) whirl mappings in three dimensions with an interesting multiplicity result in two dimensions.

Theorem 3.2.3. ($N = 2$) *For each* $k \in \mathbb{Z}$ *there exists a whirl mapping* $u_k = \mathrm{Q}[g_k]x$ *whose angle of rotation function* g_k *is given by*

$$g_k(r) = \frac{2\pi a^2 b^2 k}{b^2 - a^2}\left(\frac{1}{a^2} - \frac{1}{r^2}\right) \qquad a \le r \le b. \tag{3.2.26}$$

The mappings u_k *are equilibria of the elastic energy* (3.1.1) *with* $W(\mathbf{F}) = tr(\mathbf{F}^t\mathbf{F})/2$ *over* $\mathcal{A}(\mathbf{X})$*; specifically, for a suitable hydrostatic pressure* $\mathfrak{p}_k$*, the pair* $(u_k, \mathfrak{p}_k)$ *is a solution to the system* (3.2.1)*.*

Proof. Fix k and for the ease of notation put $u = u_k$ and $g = g_k$ and note that when $N = 2$ we have $\rho = \sqrt{x_1^2 + x_2^2} = r$. To justify the assertion first observe that

$$\nabla u(x) = \mathrm{Q}[g](r) + \frac{\dot g(r)}{r}\dot{\mathrm{Q}}[g](r)x \otimes x, \tag{3.2.27}$$

$$\Delta u(x) = \left(\frac{r\ddot g + 3\dot g}{r}\right)\dot{\mathrm{Q}}[g](r)x + \dot g^2\ddot{\mathrm{Q}}[g](r)x. \tag{3.2.28}$$

Hence referring to (3.2.3) and using the above we have

$$\nabla\mathfrak{p} = (\nabla u)^t\Delta u = \left\{(r\ddot g + 3\dot g)\left[\dot g\mathbf{I}_2 + \mathrm{Q}^t\dot{\mathrm{Q}}\right] + \dot g^2\mathrm{Q}^t\ddot{\mathrm{Q}} + \frac{\dot g^3}{r}\langle\dot{\mathrm{Q}}x, \ddot{\mathrm{Q}}x\rangle\right\}x, \tag{3.2.29}$$

that in view of the orthogonality relation $\langle\dot{\mathrm{Q}}x, \ddot{\mathrm{Q}}x\rangle = 0$ results in the equation

$$\nabla\mathfrak{p} = (\nabla u)^t\Delta u = \begin{bmatrix} -r\dot g^2\cos\theta + (3\dot g + r\ddot g)(r\dot g\cos\theta - \sin\theta) \\ -r\dot g^2\sin\theta + (3\dot g + r\ddot g)(r\dot g\sin\theta + \cos\theta)\end{bmatrix}. \tag{3.2.30}$$

Now referring to the explicit form of the angle of rotation function g as given a basic differentiation leads to

$$\dot g(r) = \frac{4\pi a^2b^2k}{b^2 - a^2}\frac{1}{r^3}, \qquad \ddot g(r) = -\frac{12\pi a^2b^2k}{b^2 - a^2}\frac{1}{r^4}, \tag{3.2.31}$$

and so a straightforward calculation results in $3\dot g_k + r\ddot g_k = 0$. As a consequence (3.2.30) simplifies and can be written as

$$\begin{aligned}(\nabla u)^t\Delta u &= \left\{(r\ddot g + 3\dot g)\left[\dot g\mathbf{I}_2 + \mathrm{Q}^t\dot{\mathrm{Q}}\right] + \dot g^2\mathrm{Q}^t\ddot{\mathrm{Q}}\right\}x \\ &= \begin{bmatrix} -r\dot g^2\cos\theta + (3\dot g + r\ddot g)(r\dot g\cos\theta - \sin\theta) \\ -r\dot g^2\sin\theta + (3\dot g + r\ddot g)(r\dot g\sin\theta + \cos\theta)\end{bmatrix} \\ &= \begin{bmatrix} -r\dot g^2\cos\theta \\ -r\dot g^2\sin\theta\end{bmatrix} = \frac{16\pi^2a^4b^4k^2}{(b^2 - a^2)^2}x|x|^{-6} \\ &= -\frac{4\pi^2a^4b^4k}{(b^2 - a^2)^2}\nabla|x|^{-4} = \nabla\mathfrak{p}.\end{aligned} \tag{3.2.32}$$

Therefore the whirl mapping u is a solution to the system (3.2.1) for a suitable choice of the hydrostatic pressure $\mathfrak{p}$. It is interesting to note that here the whirl mapping u is *totally* rotationally symmetric in that for every $\mathbf{R} \in \mathbf{SO}(2)$ we have

$$[\mathbf{R}u \circ \mathbf{R}^t](x) = \mathbf{R}u(\mathbf{R}^t x) = \mathbf{R}\mathrm{Q}[g](r)\mathbf{R}^t x = \mathbf{R}\mathbf{R}^t\mathrm{Q}[g](r)x = \mathrm{Q}[g](r)x = u(x),$$

in view of the rotation group $\mathbf{SO}(2)$ being commutative. Additionally the energy of the whirl mapping $u = u_k$ here is seen to be

$$\mathbb{E}[u;\mathbf{X}] - |\mathbf{X}| = \int_{\mathbf{X}} \frac{|\nabla u|^2 - 2}{2} = \frac{16\pi^3 a^4 b^4 k^2}{(b^2-a^2)^2}\int_a^b \frac{dr}{r^3} = \frac{8\pi^3 a^2 b^2 k^2}{b^2 - a^2}, \tag{3.2.33}$$

showing that the elastic energy of u_k diverges to infinity like k^2 as $|k| \nearrow \infty$. □

3.3 Infinitely many incompressible plane local energy minimisers

In this section we focus on the planar case $N = 2$ and show that under fairly mild assumptions on the stored energy function $W = W(\mathbf{F})$ the elastic energy (3.1.1) admits countably many L^1-local minimisers in $\mathcal{A}(\mathbf{X})$. The assumptions on W here are to be polyconvex, i.e., $W(\mathbf{F}) = \varphi(\mathbf{F}, \det \mathbf{F})$ for some convex function φ, and to be continuous, finite-valued and coercive in the sense:

$$c_0|\mathbf{F}|^2 + c_1 \le W(\mathbf{F}), \qquad \mathbf{F} \in \mathbb{R}^{2\times 2} : \det \mathbf{F} = 1, \tag{3.3.1}$$

for suitable constants $c_0, c_1, \in \mathbb{R}$ with $c_0 > 0$.

Let us begin by extending the elastic energy functional $\mathbb{E}$ to $L^1(\mathbf{X}, \mathbb{R}^2)$ by setting $\mathbb{E} \equiv \infty$ off $\mathcal{A}(\mathbf{X})$, that is, let $\mathscr{E} : L^1(\mathbf{X}, \mathbb{R}^2) \to \mathbb{R} \cup \{\infty\}$ be defined as

$$\mathscr{E}[u] = \begin{cases} \mathbb{E}[u] & u \in \mathcal{A}(\mathbf{X}) \\ \infty & \text{otherwise.} \end{cases} \tag{3.3.2}$$

Below we formulate some of the basic properties of this extended energy functional, listed here as (P1)-(P5), that will then allow us to establish the existence of an infinite family of L^1-local minimisers for $\mathbb{E}$ in $\mathcal{A}(\mathbf{X})$. Recall that $\bar{u} \in \mathcal{A}(\mathbf{X})$ is said to be an L^1-local minimiser of $\mathbb{E}$ *iff* there exists $\delta = \delta[\bar{u}] > 0$ such that for all $u \in \mathcal{A}(\mathbf{X})$:

$$||u - \bar{u}||_{L^1} < \delta \implies \mathbb{E}[\bar{u}] \le \mathbb{E}[u]. \tag{3.3.3}$$

Due to the sequential weak lower semicontinuity and coercivity of the elastic energy $\mathbb{E}$ on $W^{1,2}$ the following two properties are satisfied by the extended functional:

(P1) $\mathscr{E}$ is bounded from below and sequentially lower semicontinuous in L^1.

(P2) For each $t \in \mathbb{R}$ the level set

$$\mathscr{E}^{-1}(t) = \{u \in L^1(\mathbf{X}, \mathbb{R}^2) : \mathscr{E}[u] \leq t\}, \tag{3.3.4}$$

is closed with respect to the metric topology of L^1.

Proposition 3.3.1. *There exists an infinite family of pairwise disjoint homotopy classes $\{\mathscr{A}_k(\mathbf{X})\}_{k\in\mathbb{Z}}$ such that*

$$\mathcal{A}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{A}_k(\mathbf{X}). \tag{3.3.5}$$

Furthermore for each fixed $t \in \mathbb{R}$ and $k \in \mathbb{Z}$ the following properties are satisfied:

(P3) *The level set $\mathscr{E}^{-1}(t) \cap \mathscr{A}_k(\mathbf{X})$ is closed in $L^1(\mathbf{X}, \mathbb{R}^2)$.*

(P4) *If $(u_n : n \in \mathbb{N}) \subset \mathscr{E}^{-1}(t)$ and $\|u_n - u\|_1 \to 0$ for some $u \in \mathscr{A}_k(\mathbf{X})$ then $\exists N > 0$ such that $u_n \in \mathscr{A}_k(\mathbf{X})$ for all $n \geq N$.*

(P5) *There is an element in $\mathscr{A}_k(\mathbf{X})$ with finite $\mathscr{E}$-energy, i.e., $\exists v_k \in \mathscr{A}_k(\mathbf{X})$ such that $\mathscr{E}[v_k] < \infty$.*

Proof. Let $\mathscr{C}(\mathbf{X})$ denote the class of all continuous self-mappings of the annulus onto itself agreeing with the identity on the boundary, that is,

$$\mathscr{C}(\mathbf{X}) = \{u \in \mathbf{C}(\overline{\mathbf{X}}, \mathbb{R}^2) : u(\overline{\mathbf{X}}) \subset \overline{\mathbf{X}},\ u \equiv x \text{ on } \partial\mathbf{X}\}. \tag{3.3.6}$$

Now any admissible $u \in \mathcal{A}(\mathbf{X})$ has a continuous representative (also denoted by u) satisfying $u \in \mathscr{C}(\mathbf{X})$. To see this observe firstly that $\det \nabla u = 1$ *a.e.* combined with Lebesgue-type monotonicity arguments as, e.g., in [35] implies that u has a continuous representative $u \in \mathbf{C}(\overline{\mathbf{X}}; \mathbb{R}^2)$. Next the identity boundary condition on u and a degree theoretical argument gives that $u^{-1}(x)$ is non-empty for any $x \in \overline{\mathbf{X}}$ and therefore $\overline{\mathbf{X}} \subset u(\overline{\mathbf{X}})$. (Due to $u \equiv x$ on $\partial\mathbf{X}$ we have $d(u, \mathbf{X}, p) = 1$ for $p \in \mathbf{X}$ and $d(u, \mathbf{X}, p) = 0$ for $p \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$ where d here stands for the Brouwer degree of $u \in \mathbf{C}(\overline{\mathbf{X}}; \mathbb{R}^2)$.) Finally to justify $u(\overline{\mathbf{X}}) \subset \overline{\mathbf{X}}$ one can argue by contradiction: Suppose there exists $x \in \mathbf{X}$ such that $u(x) \notin \mathbf{X}$; then the continuity of u and Lemma 2.4 in [33] contradicts $d(u, \mathbf{X}, p) = 0$ for p outside $\overline{\mathbf{X}}$. Now we write

$$\mathscr{C}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{C}_k(\mathbf{X}), \qquad \mathscr{C}_k(\mathbf{X}) = \{u \in \mathscr{C}(\mathbf{X}) : \mathbf{deg}(u) = k\}, \tag{3.3.7}$$

where $\mathbf{deg}(u)$ here stands for the winding number of the continuous plane curve

$$\gamma^\theta(r) = \frac{u}{|u|}(r,\theta) \to \mathbb{S}^1. \tag{3.3.8}$$

Here $a \leq r = \sqrt{x_1^2 + x_2^2} \leq b$ and $\theta \in [0, 2\pi)$ fixed denote polar co-ordinates in the plane. Note that the winding number of γ^θ is independent of the choice of θ since u is continuous. Therefore we can define $\mathscr{A}_k(\mathbf{X})$ as the class of all those admissible mappings u whose continuous representative lies in $\mathscr{C}_k(\mathbf{X})$, that is, with a slight abuse of notation,

$$\mathscr{A}_k(\mathbf{X}) := \{u \in \mathcal{A}(\mathbf{X}) : \mathbf{deg}(u) = k\}, \tag{3.3.9}$$

where $\mathbf{deg}(u)$ is the winding number as above associated with the continuous representative of $u \in \mathcal{A}(\mathbf{X})$. Hence (3.3.7) gives a partition

$$\mathcal{A}(\mathbf{X}) = \bigcup_{k \in \mathbb{Z}} \mathscr{A}_k(\mathbf{X}), \tag{3.3.10}$$

and this proves (3.3.5).

It therefore remains to prove (P3)-(P5). To prove (P3) suppose that $\mathbb{E}^{-1}[t] \cap \mathscr{A}_k(\mathbf{X})$ is non-empty and take any sequence $(u_n : n \in \mathbb{N}) \subset \mathbb{E}^{-1}[t] \cap \mathscr{A}_k(\mathbf{X})$ such that $\|u_n - u\|_1 \to 0$ for some $u \in L^1(\mathbf{X}, \mathbb{R}^2)$. Then by (3.3.1) (u_n) is bounded in the $W^{1,2}$-norm and so there is a subsequence (not re-labelled) such that $u_n \rightharpoonup u$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$. Thus $u \in W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ and $u \equiv x$ on $\partial\mathbf{X}$ in the sense of traces. Moreover by a classical result of Y. Reshetnyak

$$\det \nabla u_n \overset{\star}{\rightharpoonup} \det \nabla u, \tag{3.3.11}$$

in the sense of measures and so $\det \nabla u = 1$ *a.e.*; thus $u \in \mathcal{A}(\mathbf{X})$. Furthermore as $\det \nabla u_n = 1$ *a.e.* it is known that the sequence of continuous representatives (u_n) are equi-continuous and uniformly bounded in $\mathscr{C}(\mathbf{X})$ and therefore, by Arzela-Ascoli's compactness theorem, upon extraction of a further subsequence $u_n \to u$ uniformly in $\mathscr{C}(\mathbf{X})$. Thus $\mathbf{deg}(u_n) \to \mathbf{deg}(u)$ which in particular implies $u \in \mathscr{A}_k(\mathbf{X})$ and the sequential lower semicontinuity of $\mathbb{E}$ gives $u \in \mathbb{E}^{-1}(t) \cap \mathscr{A}_k(\mathbf{X})$. This therefore gives (P3).

The proof of (P4) follows a similar pattern. Indeed $(u_n : n \in \mathbb{N}) \subset \mathbb{E}^{-1}[t]$ by (3.3.1) is bounded in the $W^{1,2}$-norm so passing to a subsequence (not re-labelled) $u_n \rightharpoonup u$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ and uniformly in $\mathscr{C}(\mathbf{X})$; hence $\mathbf{deg}(u_n) \to \mathbf{deg}(u) = k$. However as $\mathbf{deg}$ is integer-valued the latter convergence implies that $\exists N > 0$ such that $\mathbf{deg}(u_n) = k$ for $n \geq N$ and therefore $(u_n : n \geq N) \subset \mathscr{A}_k(\mathbf{X})$ which gives (P4). Finally to justify (P5) it suffices to consider for each $k \in \mathbb{Z}$ fixed the mapping $v_k(x) = \mathrm{Q}[g_k]x$ where,

$$\mathrm{Q}[g_k] = \begin{bmatrix} \cos g_k & -\sin g_k \\ \sin g_k & \cos g_k \end{bmatrix}, \quad \text{and} \quad g_k(r) = \frac{2\pi k}{b-a}(r-a). \tag{3.3.12}$$

It is easily seen that $v_k \in \mathscr{A}_k(\mathbf{X})$ and by the boundedness of W on bounded sets in $\{\mathbf{F} \in \mathbb{R}^{2\times 2} : \det \mathbf{F} = 1\}$ (as a result of continuity) we have $\mathbb{E}[v_k] < \infty$. This therefore completes the proof. $\square$

Now combining the previous proposition with (P1) and (P2) we are able to prove the existence of countably many L^1 local minimiser for the elastic energy $\mathbb{E}$.

Theorem 3.3.2. *For each $k \in \mathbb{Z}$ there exists an admissible mapping $u_k \in \mathscr{A}_k(\mathbf{X})$ serving as an L^1-local minimiser of $\mathbb{E}$ over $\mathcal{A}(\mathbf{X})$ characterised as the minimiser*

$$\mathbb{E}[u_k; \mathbf{X}] = \inf_{v \in \mathscr{A}_k(\mathbf{X})} \mathbb{E}[v; \mathbf{X}]. \tag{3.3.13}$$

In particular $(u_k : k \in \mathbb{Z}) \subset \mathcal{A}(\mathbf{X})$ is a countable family of L^1-local minimiser of $\mathbb{E}$ over $\mathcal{A}(\mathbf{X})$.

Proof. The existence of a minimiser with finite energy in each homotopy class $\mathscr{A}_k(\mathbf{X})$ is easy and shall not be given. Let $u_k \in \mathscr{A}_k(\mathbf{X})$ denote this minimiser. Using (P4) we show that u_k is a L^1-local minimiser of $\mathbb{E}$ in $\mathcal{A}(\mathbf{X})$. Indeed by way of contradiction suppose that $w := u_k$ is not a L^1-local minimiser; then $\exists (w_n : n \in \mathbb{N}) \subset \mathcal{A}(\mathbf{X})$ such that $||w_n - w||_{L^1} \to 0$ as $n \nearrow \infty$ whilst $\mathbb{E}[w_n] < \mathbb{E}[w]$ for every $n \in \mathbb{N}$. This however by (P4) implies that $\exists N > 0$ such that $w_n \in \mathscr{A}_k(\mathbf{X})$ for $n \geq N$ which is a contradiction as $\mathbb{E}[w_n] < \mathbb{E}[w] = \mathbb{E}[u_k] = \inf_{u \in \mathscr{A}_k(\mathbf{X})} \mathbb{E}$. Hence u_k must be an L^1-local minimiser as claimed. $\square$

We now turn our attention to the distances between the homotopy classes $\mathscr{A}_k(\mathbf{X})$ in various L^p-norms for when $N = 2$. Given a pair $\mathscr{A}_k$ and $\mathscr{A}_m$ $(m \neq k)$ we compute the distance between them in the usual Lebesgue norms: $\sup_{x \in \mathbf{X}} |\cdot|$ and $\|\cdot\|_p$. The following statement gives the distance between homotopy classes in the uniform or L^∞-norm.

Proposition 3.3.3. *For any pair of homotopy classes $\mathscr{A}_k(\mathbf{X})$ and $\mathscr{A}_m(\mathbf{X})$ with $k \neq m$ the distance with respect to the uniform norm is given by*

$$\delta_U(k,m) := \inf_{\substack{u \in \mathscr{A}_k(\mathbf{X}) \\ v \in \mathscr{A}_m(\mathbf{X})}} \sup_{x \in \mathbf{X}} |u(x) - v(x)| = 2a. \tag{3.3.14}$$

Proof. Firstly if we restrict to the subclass of whirl mapping then it is evident that we have the upper bound

$$\delta_U(k,m) \leq \inf_{\substack{u \in \mathscr{A}_k(\mathbf{X}) \\ v \in \mathscr{A}_m(\mathbf{X}) \\ u,v \text{ whirls}}} \sup_{x \in \mathbf{X}} |u(x) - v(x)|. \tag{3.3.15}$$

Hence upon referring to the specific representation of whirl mappings we have that

$$||u - v||_{L^\infty(\mathbf{X};\mathbb{R}^2)} = \sup_{x\in\mathbf{X}}|u(x) - v(x)| = \sup_{x\in\mathbf{X}} r|(\mathrm{Q}[g_1] - \mathrm{Q}[g_2])x|x|^{-1}|. \tag{3.3.16}$$

Now basic calculation gives $|(\mathrm{Q}[g_1] - \mathrm{Q}[g_2])x|x|^{-1}| = \sqrt{2\,(1 - \cos(g_1 - g_2))}$ and therefore

$$\begin{aligned} ||u - v||_{L^\infty(\mathbf{X};\mathbb{R}^2)} &= \sup_{x\in\mathbf{X}}|u(x) - v(x)| \\ &= \sqrt{2} \sup_{r\in[a,b]} r\sqrt{(1 - \cos(g_1 - g_2))}. \end{aligned} \tag{3.3.17}$$

Next suppose that we further consider the particular whirls u, v with angle of rotation functions

$$g_1(r) = \begin{cases} 0 & r \in [a, a+\varepsilon], \\ 2\pi k(r - a - \varepsilon)(b - a - \varepsilon)^{-1} & r \in [a+\varepsilon, b], \end{cases} \tag{3.3.18}$$

$$g_2(r) = \begin{cases} 2\pi(m - k)(r - a - \varepsilon)\varepsilon^{-1} & r \in [a, a+\varepsilon], \\ g_1(r) & r \in [a+\varepsilon, b], \end{cases} \tag{3.3.19}$$

where $0 < \varepsilon < b - a$. Since $\sqrt{1 - \cos[2\pi(m - k)\varepsilon^{-1}(r - a - \varepsilon)]} \le \sqrt{2}$ for $a \le r \le a + \varepsilon$ we conclude that

$$||u - v||_{L^\infty(\mathbf{X};\mathbb{R}^2)} \le 2(a + \varepsilon), \tag{3.3.20}$$

where u and v are whirl mappings with angle of rotation functions g_1, g_2 given by (3.3.18) and (3.3.19) respectively. (One can easily check using the winding number of the curves $u/|u|(r)$ and $v/|v|(r)$ that $\mathbf{deg}(u) = k$ and $\mathbf{deg}(v) = m$ and so $u \in \mathscr{A}_k$, $v \in \mathscr{A}_m$.) Thus,

$$\delta_U(k, m) \le 2(a + \varepsilon) \tag{3.3.21}$$

for all $\varepsilon \in (0, b - a]$ and hence $\delta_U(k, m) \le 2a$. Now to obtain a lower bound we first note that,

$$\begin{aligned} |u(x) - v(x)| &\ge \min\{|u|, |v|\}||u|^{-1}u(x) - |v|^{-1}v(x)| \\ &\ge a||u|^{-1}u(x) - |v|^{-1}v(x)|. \end{aligned} \tag{3.3.22}$$

Moreover since the maps $|u|^{-1}u$ and $|v|^{-1}v$ are $\mathbb{S}^1$ valued we obtain that,

$$\sup_{x\in\mathbf{X}}|u(x) - v(x)| \ge \sup_{x\in\mathbf{X}} a\left||u|^{-1}u(x) - |v|^{-1}v(x)\right| = 2a. \tag{3.3.23}$$

Note that the last equality comes from the fact that for each fixed $\theta \in (0, 2\pi)$ the continuous $\mathbb{S}^1$ valued curves $u/|u|(r, \theta)$ and $v/|v|(r, \theta)$ for $r \in (a, b)$ have degrees k and m respectively. Therefore as $k \ne m$ there exists at least one point $c \in (a, b)$ such that $u/|u|(c, \theta) = -v/|v|(c, \theta)$ and thus $\exists x \in \mathbf{X}$ such that $||u|^{-1}u(x) - |v|^{-1}v(x)| = 2$. Therefore $\delta_U(k, m) = 2a$ for each pair of distinct integers $k, m \in \mathbb{Z}$. $\square$

For the remaining L^p-norms we can again calculate the distance between homotopy classes but with a completely different outcome as is described below.

Proposition 3.3.4. *For $k, m \in \mathbb{Z}$ and $1 \leq p < \infty$ the corresponding L^p distance between the homotopy classes $\mathscr{A}_k(\mathbf{X})$ and $\mathscr{A}_m(\mathbf{X})$ is given by,*

$$\delta_p(k,m) = \inf_{\substack{u \in \mathscr{A}_k(\mathbf{X}) \\ v \in \mathscr{A}_m(\mathbf{X})}} \|u - v\|_p = 0. \tag{3.3.24}$$

Proof. As in the uniform distance we can bound this above by restricting to the class of whirl maps with the homotopy classes. So,

$$\delta_p(k,m) \leq \inf_{\substack{u \in \mathscr{A}_k(\mathbf{X}) \\ v \in \mathscr{A}_m(\mathbf{X}) \\ u,v \text{whirls}}} \|u - v\|_p. \tag{3.3.25}$$

Hence, the L^p-distance between a pair of whirls u, v we can easily calculate that,

$$\|u - v\|_p^p = \int_{\mathbf{X}} |(\mathrm{Q}[g_1] - \mathrm{Q}[g_2])x|^p \, dx = \int_{\mathbf{X}} 2^{\frac{p}{2}} |1 - \cos(g_1 - g_2)|^{\frac{p}{2}} |x|^p \, dx. \tag{3.3.26}$$

Now we restrict to the simpler case when $g_1 = 2\pi k f_\varepsilon$ and $g_2 = 2\pi m f_\varepsilon$ where f_ε is defined by,

$$f_\varepsilon(r) = \begin{cases} 0 & r \in [a, b - \varepsilon], \\ (b - r + \varepsilon)/\varepsilon & r \in [b - \varepsilon, b]. \end{cases} \tag{3.3.27}$$

It can be easily checked that the $\mathbf{deg}(u) = k$ and $\mathbf{deg}(v) = m$. Moreover the L^p distance between these two whirls is,

$$\begin{aligned} \|u - v\|_p^p &= \int_0^{2\pi} \int_{b-\varepsilon}^{b} 2^{\frac{p}{2}} |1 - \cos\left(2\pi(k - m)(b - r + \varepsilon)/\varepsilon\right)|^{\frac{p}{2}} r^{p+1} \, dr d\theta \\ &\leq \int_0^{2\pi} \int_{b-\varepsilon}^{b} 2^p r^{p+1} \, dr d\theta = 2\pi \frac{2^p}{p+2} \left(b^{p+2} - (b - \varepsilon)^{p+2}\right). \end{aligned} \tag{3.3.28}$$

Therefore $\|u - v\|_p$ can be made as small as we like and so the sought L^p distance is zero, i.e., $\delta_p(k, m) = 0$ for $1 \leq p < \infty$. □

3.4 Structure of whirl mappings in higher dimensions

In the remainder of this paper we show how the concepts of whirl mappings and their symmetries can be extended to higher dimensions and investigate whether these class of mappings provide equilibria for the Dirichlet energy over the space $\mathcal{A}(\mathbf{X})$.

Towards this end let us start by defining a generalised whirl mapping u of an annulus $\mathbf{X} = \mathbf{X}[a, b] \subset \mathbb{R}^N$ as a continuous self-mapping of $\overline{\mathbf{X}}$ onto itself agreeing with the identity on $\partial\mathbf{X}$ and having the specific representation

$$u : x \mapsto u(x) = \begin{cases} \mathrm{Q}(\rho_1, \dots, \rho_{n-1}, \rho_n)\, x, & \text{if } N = 2n, \\ \mathrm{Q}(\rho_1, \dots, \rho_{n-1}, x_N)\, x, & \text{if } N = 2n-1. \end{cases} \tag{3.4.1}$$

Here $x = (x_1, ..., x_N) \in \mathbf{X}$ and for $1 \le j \le n$ when $N = 2n$ and $1 \le j \le n-1$ when $N = 2n-1$ we set $\rho_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2}$ while Q is a suitable mapping (*see* below) taking values in $\mathbf{SO}(N)$.

For the ease of notation we agree to set, when $N = 2n-1$, $\rho_n = x_N$, so that as a result we can write u in (3.4.1), regardless of N being even or odd, as

$$u : x \mapsto u(x) = \mathrm{Q}(\rho_1, \dots, \rho_n)x, \qquad x \in \overline{\mathbf{X}}. \tag{3.4.2}$$

With this notation in place we now require Q to lie in $\mathbf{C}(\overline{U}_N, \mathbb{T})$ where $\mathbb{T}$ is the fixed maximal torus in $\mathbf{SO}(N)$ of all block diagonal 2×2 rotation matrices and

$$U_N = \begin{cases} \{\varrho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n_+ : a < |\varrho| < b\}, & \text{if } N = 2n, \\ \{\varrho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^{n-1}_+ \times \mathbb{R} : a < |\varrho| < b\}, & \text{if } N = 2n-1. \end{cases}$$

As a result when $N = 2n-1$ we have $\mathrm{Q}(\varrho) = \mathrm{diag}(\mathbf{R}[g_1](\varrho), \dots, \mathbf{R}[g_n](\varrho), 1)$ or more specifically

$$\mathrm{Q}(\varrho) = \mathrm{Q}(\rho_1, \cdots, \rho_n) = \begin{bmatrix} \mathbf{R}[g_1] & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & \mathbf{R}[g_{n-1}] & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{3.4.3}$$

and when $N = 2n$ then $\mathrm{Q}(\varrho) = \mathrm{diag}(\mathbf{R}[g_1](\varrho), \dots, \mathbf{R}[g_n](\varrho))$, that is,

$$\mathrm{Q}(\varrho) = \mathrm{Q}(\rho_1, \cdots \rho_n) = \begin{bmatrix} \mathbf{R}[g_1] & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & \mathbf{R}[g_{n-1}] & 0 \\ 0 & 0 & \cdots & \mathbf{R}[g_n] \end{bmatrix} \tag{3.4.4}$$

where in either case the angle of rotation or whirl functions $g_j = g_j(\varrho) : \overline{U}_N \to \mathbb{R}$ and we have denoted

$$\mathbf{R}[g_j](\varrho) = \begin{bmatrix} \cos g_j(\varrho) & -\sin g_j(\varrho) \\ \sin g_j(\varrho) & \cos g_j(\varrho) \end{bmatrix}. \tag{3.4.5}$$

Let us now proceed, upon assuming sufficient differentiability on the functions g_j, with some direct calculations and for the sake of future reference collect some useful identities.

Firstly we have

$$\nabla u = \mathrm{Q} + \sum_{j=1}^{n} \mathrm{Q}_{,j} x \otimes \nabla \rho_j, \tag{3.4.6}$$

which by taking the Hilbert-Schmidt norm for the sake of the energy integrand in turn gives the identity,

$$|\nabla u|^2 = tr\Big\{ \mathrm{I}_N + \sum_{j=1}^{n} [\mathrm{Q}_{,j} x \otimes \mathrm{Q}\nabla\rho_j + \mathrm{Q}\nabla\rho_j \otimes \mathrm{Q}_{,j} x] + \sum_{j=1}^{n} \mathrm{Q}_{,j} x \otimes \mathrm{Q}_{,j} x \Big\}. \tag{3.4.7}$$

Here $\mathrm{Q}_{,j}$ denotes the partial derivative of Q with respect to the ρ_j variable. Next we show, upon utilising (3.4.6), that whirls mappings satisfy the incompressibility constraint. Indeed by the above calculation

$$\det \nabla u = \det \left(\mathrm{I}_N + \sum_{j=1}^{n} \mathrm{Q}^t \mathrm{Q}_{,j} x \otimes \nabla \rho_j \right). \tag{3.4.8}$$

Now we note that,

$$\mathrm{Q}^t \mathrm{Q}_{,j} = \begin{cases} \mathrm{diag}(\partial_j g_1 \mathbf{J}, \ldots, \partial_j g_{n-1} \mathbf{J}, 0) & \text{if } N = 2n-1, \\ \mathrm{diag}(\partial_j g_1 \mathbf{J}, \ldots, \partial_j g_{n-1} \mathbf{J}, \partial_j g_n \mathbf{J}) & \text{if } N = 2n. \end{cases} \tag{3.4.9}$$

where in the above for short we have set $\partial_j g_i = \partial_{\rho_j} g_i$ while

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.4.10}$$

Next let $y_j = (x_{2j-1}, x_{2j})$ for $1 \le j \le n$ if $N = 2n$ and $y_n = x_{2n-1}$ when $N = 2n-1$. Then it is easily seen that we have

$$\mathrm{Q}^t \mathrm{Q}_{,j} x = \begin{cases} (\partial_j g_1 \mathbf{J} y_1, \ldots, \partial_j g_{n-1} \mathbf{J} y_{n-1}, 0)^t & \text{if } N = 2n-1, \\ (\partial_j g_1 \mathbf{J} y_1, \ldots, \partial_j g_{n-1} \mathbf{J} y_{n-1}, \partial_j g_n \mathbf{J} y_n)^t & \text{if } N = 2n. \end{cases} \tag{3.4.11}$$

Furthermore from the definition of ρ_j it is clear that $\nabla \rho_j = (0, \ldots, y_j/\rho_j, \ldots, 0)$ and therefore in view of $\mathbf{J}$ being skew-symmetric it is plain that

$$\langle \mathrm{Q}^t \mathrm{Q}_{,j} x, \nabla \rho_i \rangle = \langle \frac{\partial g_i}{\partial \rho_j} \mathbf{J} y_i, \frac{y_i}{\rho_i} \rangle = 0, \tag{3.4.12}$$

for all $1 \le i, j \le n$. With this observation in mind we now state a lemma which will give us that our whirl mappings are incompressible.

Lemma 3.4.1. *Suppose that $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ are two strings of vectors in $\mathbb{R}^n$ satisfying the orthogonality relations $\langle a_i, b_j \rangle = 0$ for each $1 \le i, j \le k$. Then*

$$\det \left(\mathrm{I}_n + \sum_{j=1}^{k} a_j \otimes b_j \right) = 1. \tag{3.4.13}$$

Proof. This is by induction on k. Firstly when $k = 1$ for the rank-one perturbation of I_n we have

$$\det\left(\mathrm{I}_n + a_1 \otimes b_1\right) = 1 + \langle a_1, b_1 \rangle = 1. \tag{3.4.14}$$

Before proceeding to the case of an arbitrary $k \geq 2$ it is instructive to see how the case $k = 2$ works. Towards this end let $A_1 = \mathrm{I}_n + a_1 \otimes b_1$ and note that $A_1^{-1} = \mathrm{I}_n - a_1 \otimes b_1$. Then

$$\det(\mathrm{I}_n + a_1 \otimes b_1 + a_2 \otimes b_2) = \left(1 + \langle b_2, A_1^{-1} a_2 \rangle\right) \det A_1.$$

However since $\det A_1 = 1$ and $A_1^{-1} a_2 = a_2 - a_1 \langle a_2, b_1 \rangle = a_2$ it follows that

$$\det(\mathrm{I}_n + a_1 \otimes b_1 + a_2 \otimes b_2) = (1 + \langle b_2, a_2 \rangle) = 1.$$

Thus we have shown that (3.4.13) holds in the case that $k = 2$. Let us now assume that (3.4.13) holds for a fixed k. Here we define A_k to be

$$A_k = \mathrm{I}_n + \sum_{j=1}^{k} a_j \otimes b_j, \tag{3.4.15}$$

and again here it can be easily verified that $A_k^{-1} = \mathrm{I}_n - \sum_{j=1}^{k} a_j \otimes b_j$. Thus

$$\det\left(\mathrm{I}_n + \sum_{j=1}^{k+1} a_j \otimes b_j\right) = \left(1 + \langle b_{k+1}, A_k^{-1} a_{k+1} \rangle\right) \det A_k. \tag{3.4.16}$$

Now by our inductive hypothesis we know that $\det A_k = 1$ and by our assumption on the vectors a_j and b_j we obtain that $A_k^{-1} a_{k+1} = a_{k+1}$. Therefore

$$\det\left(\mathrm{I}_n + \sum_{j=1}^{k+1} a_j \otimes b_j\right) = (1 + \langle b_{k+1}, a_{k+1} \rangle) = 1, \tag{3.4.17}$$

which is the required conclusion for $k + 1$. This therefore completes the proof. □

Now taking $a_j = \mathrm{Q}^t \mathrm{Q}_{,j} x$ and $b_j = \nabla \rho_j$ the observation (3.4.12) combined with the conclusion in Lemma 3.4.1 gives

$$\det \nabla u = \det\left(\mathrm{I}_N + \sum_{j=1}^{n} \mathrm{Q}^t \mathrm{Q}_{,j} x \otimes \nabla \rho_j\right) = 1. \tag{3.4.18}$$

Hence the whirl mapping $u \in \mathcal{A}(\mathbf{X})$ provided that $\mathrm{Q}(\rho) = \mathrm{I}_N$ for $\rho \in \partial U_a$ and $\rho \in \partial U_b$ and therefore we shall further impose these assumptions on Q, via the angle of rotation functions g_j's. Now recalling (3.4.7) we obtain

$$|\nabla u|^2 = N + \sum_{j=1}^{n} |\mathrm{Q}_{,j} x|^2 + \sum_{j=1}^{n} tr\left(\mathrm{Q}_{,j} x \otimes \mathrm{Q} \nabla \rho_j + \mathrm{Q} \nabla \rho_j \otimes \mathrm{Q}_{,j} x\right). \tag{3.4.19}$$

However since we have

$$tr\,(\mathrm{Q}_{,j}x \otimes \mathrm{Q}\nabla\rho_j) = tr\,(\mathrm{Q}\nabla\rho_j \otimes \mathrm{Q}_{,j}x) = \langle \mathrm{Q}\nabla\rho_j, \mathrm{Q}_{,j}x\rangle = \langle \nabla\rho_j, \mathrm{Q}^t\mathrm{Q}_{,j}x\rangle = 0,$$

it follows that the third term in (3.4.19) vanishes and therefore the Hilbert-Schmidt norm of the gradient is given by

$$|\nabla u|^2 = N + \sum_{j=1}^{n} |\mathrm{Q}_{,j}x|^2. \tag{3.4.20}$$

Consequently the Dirichlet energy associated with a whirl mappings u results from integrating (3.4.20) and takes the form

$$\mathbb{E}[u;\mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} |\nabla u|^2 = \frac{N}{2}|\mathbf{X}| + \frac{1}{2}\int_{\mathbf{X}} \sum_{j=1}^{n} |\mathrm{Q}_{,j}x|^2\,dx.$$

Now referring to the explicit representation of Q in terms of g as expressed at the beginning of the section it is easily seen that

$$\sum_{j=1}^{n} |\mathrm{Q}_{,j}x|^2 = \sum_{j=1}^{n}\sum_{l=1}^{s} (g_{l,j})^2\rho_l^2 = \sum_{l=1}^{s} |\nabla g_l|^2\rho_l^2,$$

where $s = n$ if $N = 2n$ or $s = n-1$ if $N = 2n-1$. Therefore a change of variables gives

$$\begin{aligned}\mathbb{E}[u;\mathbf{X}] - \frac{N}{2}|\mathbf{X}| &= \frac{(2\pi)^s}{2}\int_{U_N} \sum_{l=1}^{s} |\nabla g_l|^2 \rho_k^2 \prod_{j=1}^{s} \rho_j\,d\rho \\ &= \frac{(2\pi)^s}{2}\mathbb{H}[\mathbf{g};U_N].\end{aligned}$$

Furthermore by inspection it is evident that the functional $\mathbb{H}$ can be decomposed as a finite sum

$$\mathbb{H}[\mathbf{g};U_N] = \sum_{l=1}^{s} \mathbb{H}_l[g_l;U_N], \tag{3.4.21}$$

where $\mathbf{g} = (g_1, ..., g_s)$ and each summand $\mathbb{H}_l$ has the form

$$\mathbb{H}_l[g;U_N] = \int_{U_N} |\nabla g|^2\rho_l^2 \prod_{j=1}^{s}\rho_j\,d\rho. \tag{3.4.22}$$

The terms $\mathbb{H}_l$ sum up to give the restricted energy functional $\mathbb{H}$ associated with generalised whirl mappings and as in the previous lower dimensional cases only depend on the whirl functions $\mathbf{g} = (g_1 \dots, g_s)$. However since the energy functional $\mathbb{H}$ is a sum of $\mathbb{H}_l$s and each $\mathbb{H}_l$ depends only on a single angle of rotation function $g = g_l$ in the analysis of the Euler-Lagrange equation we can focus on each $\mathbb{H}_l$ individually. Now the suitable admissible class of mappings for each $\mathbb{H}_l$ is given by

$$\mathbf{G}(U_N) = \bigcup_{k\in\mathbb{Z}} \mathbf{G}_k(U_N) \tag{3.4.23}$$

where the components in the union on the right for each fixed $k \in \mathbb{Z}$ are given by

$$\mathbf{G}_k(U_N) = \Big\{ g \in W^{1,2}(U_N) : g(\rho) = 0 \text{ if } \rho \in \partial(U_N)_a, \\ \text{and } g(\rho) = 2\pi k \text{ if } \rho \in \partial(U_N)_b \Big\}. \tag{3.4.24}$$

One can then check that the Euler-Lagrange equation associated with $\mathbb{H}_l$ over each $\mathbf{G}_k(U_N)$ to be given by

$$\begin{cases} div\left(\rho_l^2 \prod_{j=1}^s \rho_j \nabla g\right) = 0 & \varrho \in U_N, \\ g = 0 & \varrho \in (\partial U_N)_a, \\ g = 2\pi k & \varrho \in (\partial U_N)_b, \\ \rho_l^2 \prod_{j=1}^s \rho_j \partial_\nu g = 0 & \varrho \in \partial U_N \backslash [(\partial U_N)_a \cup (\partial U_N)_b] \end{cases} \tag{3.4.25}$$

The next result states that for each $k \in \mathbb{Z}$ the boundary value problem (3.4.25) has a unique solution.

Proposition 3.4.2. *For any $k \in \mathbb{Z}$ the Euler-Lagrange equations* (3.4.25) *has a unique solution $g = g(\varrho) = g(\rho_1, ..., \rho_n)$ given by,*

$$g(\rho_1, ..., \rho_n; k) = \frac{2\pi a^N b^N k}{b^N - a^N} \left(\frac{1}{a^N} - \frac{1}{\left(\sqrt{\sum_{i=1}^n \rho_i^2}\right)^N} \right). \tag{3.4.26}$$

Proof. Firstly for uniqueness suppose that g_1, g_2 are solutions to (3.4.25) and let $g = g_1 - g_2$ which will solve (3.4.25) but now with all boundary conditions zero. Then by an application of the divergence theorem we have

$$\int_{U_N} \rho_l^2 \prod_{j=1}^s \rho_j |\nabla g|^2 \, d\rho = \int_{\partial U_N} g \rho_l^2 \prod_{j=1}^s \rho_j \partial_\nu g \, d\sigma = 0, \tag{3.4.27}$$

as $g(\varrho) = 0$ for $\varrho \in (\partial U_N)_a$ and $\varrho \in (\partial U_N)_b$. Now as we have $\rho_j \geq 0$ for each $1 \leq j \leq n$ it follows that $|\nabla g| = 0$ and so $g \equiv 0$ due to the zero boundary conditions which then gives $g_1 = g_2$.

Therefore we are left to verify that for each $k \in \mathbb{Z}$ the given whirl function $g(\cdot\,; k)$ satisfies (3.4.25). To this end note that g satisfies all the required boundary conditions. Furthermore by differentiation

$$\partial_{\rho_l} g(\rho\,; k) = \frac{2\pi a^N b^N k}{b^N - a^N} \frac{N \rho_l}{\left(\sqrt{\sum_{i=1}^n \rho_i^2}\right)^{N+2}}. \tag{3.4.28}$$

Now let us proceed by considering the even and odd cases of N separately. In the former case $N = 2n$ upon using (3.4.28) we obtain that

$$
\begin{aligned}
div\left(\rho_l^2\prod_{j=1}^{n}\rho_j\nabla g\right) =& \frac{2\pi N a^N b^N k}{b^N - a^N}\sum_{r=1}^{n}\frac{\partial}{\partial\rho_r}\left(\rho_l^2\prod_{\substack{j=1\\ j\neq r}}^{n}\rho_j\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right)\\
=& \frac{2\pi N a^N b^N k}{b^N - a^N}\left\{\sum_{\substack{r=1\\ r\neq l}}^{n}\rho_l^2\prod_{\substack{j=1\\ j\neq r}}^{n}\rho_j\frac{\partial}{\partial\rho_r}\left(\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right) +\right.\\
&\left. + \left(\prod_{\substack{j=1\\ j\neq l}}^{n}\rho_j\right)\frac{\partial}{\partial\rho_l}\left(\frac{\rho_l^4}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right)\right\}\\
=& \frac{2\pi N a^N b^N k}{b^N - a^N}\left\{\sum_{\substack{r=1\\ r\neq l}}^{n}\frac{\rho_l^2\prod_{j=1,j\neq r}^{n}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\left(2\rho_r - \frac{(2n+2)\rho_r^3}{\left(\sum_{i=1}^n\rho_i^2\right)}\right) +\right.\\
&\left. + \frac{\prod_{j=1,j\neq l}^{n}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\left(4\rho_l^3 - \frac{(2n+2)\rho_l^5}{\left(\sum_{i=1}^n\rho_i^2\right)}\right)\right\}
\end{aligned}
$$

and consequently

$$
\begin{aligned}
div\left(\rho_l^2\prod_{j=1}^{n}\rho_j\nabla g\right) =& \frac{2\pi N a^N b^N k}{b^N - a^N}\left\{\sum_{r=1}^{n}\frac{\rho_l^2\prod_{j=1}^{n}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\left(2 - \frac{(2n+2)\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)}\right) +\right.\\
&\left. + 2\frac{\rho_l^2\prod_{j=1}^{n}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right\}\\
=& \frac{2\pi N a^N b^N k}{b^N - a^N}\left\{\frac{\rho_l^2\prod_{j=1}^{n}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\left(2n - (2n+2)\right) +\right.\\
&\left. + 2\frac{\rho_l^2\prod_{j=1}^{n}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right\} = 0.
\end{aligned}
$$

Thus we have that (3.4.26) solves (3.4.25) and therefore is the unique solution in the even dimensional case $N = 2n$. Now for when $N = 2n - 1$ the calculations proceed in a similar fashion but with a careful change accounting for $\rho_n = x_n$. Indeed here we have

$$
\begin{aligned}
div\left(\rho_l^2\prod_{j=1}^{n-1}\rho_j\nabla g\right) =& \frac{2\pi N a^N b^N k}{b^N - a^N}\left\{\sum_{r=1}^{n-1}\frac{\partial}{\partial\rho_r}\left(\rho_l^2\prod_{\substack{j=1\\ j\neq r}}^{n-1}\rho_j\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1/2}}\right) +\right.\\
&\left. + \rho_l^2\prod_{j=1}^{n-1}\rho_j\frac{\partial}{\partial\rho_n}\left(\frac{\rho_n}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1/2}}\right)\right\}\\
=& \frac{2\pi N a^N b^N k}{b^N - a^N}\times(\mathbf{I} + \mathbf{II}).
\end{aligned}
\tag{3.4.29}
$$

Then

$$\mathbf{I} = \frac{1}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{1/2}} \sum_{r=1}^{n-1} \frac{\partial}{\partial \rho_r} \left(\rho_l^2 \prod_{\substack{j=1 \\ j \neq r}}^{n-1} \rho_j \frac{\rho_r^2}{\left(\sum_{i=1}^{n} \rho_i^2\right)^n} \right) +$$

$$+ \sum_{r=1}^{n-1} \frac{\partial}{\partial \rho_r} \left(\frac{1}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{1/2}} \right) \left(\rho_l^2 \prod_{\substack{j=1 \\ j \neq r}}^{n-1} \rho_j \frac{\rho_r^2}{\left(\sum_{i=1}^{n} \rho_i^2\right)^n} \right) = \qquad \mathbf{I}_1 + \mathbf{I}_2, \tag{3.4.30}$$

where we have

$$\begin{aligned} \mathbf{I}_1 &= \sum_{r=1}^{n-1} \frac{\rho_l^2 \prod_{j=1}^{n-1} \rho_j}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{n+1/2}} \left(2 - \frac{2n\rho_r^2}{\left(\sum_{i=1}^{n} \rho_i^2\right)} \right) + 2 \frac{\rho_l^2 \prod_{j=1}^{n-1} \rho_j}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{n+1/2}} \\ &= \frac{\rho_l^2 \prod_{j=1}^{n-1} \rho_j}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{n+1/2}} \left(2n - 2n \frac{\sum_{i=1}^{n} \rho_i^2 - \rho_n^2}{\sum_{i=1}^{n} \rho_i^2} \right) \\ &= 2n \frac{\rho_l^2 \rho_n^2 \prod_{j=1}^{n-1} \rho_j}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{n+3/2}}, \end{aligned} \tag{3.4.31}$$

and

$$\mathbf{I}_2 = -\sum_{r=1}^{n-1} \rho_l^2 \prod_{j=1}^{n-1} \rho_j \frac{\rho_r^2}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{n+3/2}} = -\frac{\rho_l^2 \prod_{j=1}^{n-1} \rho_j}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{n+3/2}} \left(\sum_{i=1}^{n-1} \rho_i^2 \right).$$

Likewise a straightforward calculation gives

$$\mathbf{II} = \frac{\rho_l^2 \prod_{j=1}^{n-1} \rho_j}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{n+1/2}} \left(1 - \frac{(2n+1)\rho_n^2}{\sum_{i=1}^{n} \rho_i^2} \right). \tag{3.4.32}$$

Hence by putting the various fragments of the above calculations and derivations together it is seen at once that the divergence term in the set of equations can be written as

$$\begin{aligned} div \left(\rho_l^2 \prod_{j=1}^{n} \rho_j \nabla g \right) &= \frac{2\pi N a^N b^N k}{b^N - a^N} \times (\mathbf{I} + \mathbf{II}) \\ &= \frac{2\pi N a^N b^N k}{b^N - a^N} \frac{\rho_l^2 \prod_{j=1}^{n-1} \rho_j}{\left(\sum_{i=1}^{n} \rho_i^2\right)^{n+1/2}} \times \\ &\times \left(1 - \frac{(2n+1)\rho_n^2 - 2n\rho_n^2 + \sum_{r=1}^{n-1} \rho_r^2}{\sum_{i=1}^{n} \rho_i^2} \right) = 0. \end{aligned} \tag{3.4.33}$$

Therefore we have shown that (3.4.26) is also the unique solution to (3.4.25) in the odd dimensional case $N = 2n - 1$. □

Remark 3.4.1. Note that upon writing $r = \sqrt{x_1^2 + ... + x_N^2} = \sqrt{\rho_1^2 + ... + \rho_n^2}$ it is easy to detect that the function given by (3.4.26) depends on r only and that for each $k \in \mathbb{Z}$ the solution (3.4.26) to (3.4.25) only varies by a constants depending on the boundary conditions. This specifically means that for each $k \in \mathbb{Z}$ we can write (3.4.26) as

$$g(r\,; k) = d(k) - c(k) r^{-N}, \tag{3.4.34}$$

where

$$c(k) = \frac{2\pi a^N b^N k}{b^N - a^N}, \qquad d(k) = \frac{2\pi b^N k}{b^N - a^N}. \tag{3.4.35}$$

Thus any solution $\mathbf{g} = (g_1, ..., g_s)$ to the Euler-Lagrange equation associated with $\mathbb{H}[\mathbf{g}, U_N]$ depends solely on r, in fact, $\mathbf{g} = \mathbf{d} - \mathbf{c}r^{-N}$. In particular the corresponding whirl mapping is of the form $u(x) = \mathrm{Q}(r)x$ for $\mathrm{Q} \in \mathbf{C}^1([a,b], \mathbf{SO}(N))$.

3.5 Whirl mappings as solutions to the nonlinear system (3.2.1) in higher dimensions $N \geq 4$

In this final section of the paper we show that in higher dimensions $N \geq 4$ the non-trivial whirl mappings obtained through the critical points of the restricted energy can go on to satisfy the Euler-Lagrange system (3.2.1) associated with the Dirichlet energy $W(\mathbf{F}) = tr(\mathbf{F}^t\mathbf{F})/2$ only when $N = 2n$. In sharp contrast when $N = 2n - 1$ the only whirl solution to (3.2.1) is the trivial identity mapping. The conclusion is therefore similar in spirit to the cases $N = 2$ *vs.* $N = 3$ discussed earlier in the paper.

Towards this end recall that the Euler-Lagrange equation associated with the Dirichlet energy over $\mathcal{A}(\mathbf{X})$ takes the form [*cf.* (3.1.6), (3.2.1)]

$$\begin{aligned} div\, \mathfrak{S}[x, \nabla u(x)] &= \Delta u - div(\mathfrak{p}(x) \operatorname{cof} \nabla u) \\ &= \Delta u - (\operatorname{cof} \nabla u) \nabla \mathfrak{p}(x) = 0, \end{aligned} \tag{3.5.1}$$

or using the incompressibility of u and basic identities upon rearranging terms

$$(\nabla u)^t \Delta u = \nabla \mathfrak{p}. \tag{3.5.2}$$

Now straightforward calculations using the notation introduced in the previous section lead to the identities (here *dots* denote derivatives of Q with respect to r in light of Remark 3.4.1),

$$\nabla u = \mathrm{Q} + r^{-1}\dot{\mathrm{Q}}x \otimes x, \tag{3.5.3}$$

$$\Delta u = r^{-1}\left[(N+1)\dot{\mathrm{Q}} + r\ddot{\mathrm{Q}}\right] x. \tag{3.5.4}$$

Next referring to the previous section and by using the explicit form of Q and g as given by (3.4.34) we can easily verify that

$$\dot{\mathrm{Q}} = Nr^{-(N+1)}\mathrm{Q}\mathbf{C}, \tag{3.5.5}$$

$$\ddot{\mathrm{Q}} = -N(N+1)r^{-(N+2)}\mathrm{Q}\mathbf{C} + N^2 r^{-(2N+2)}\mathrm{Q}\mathbf{C}^2, \tag{3.5.6}$$

where $\mathbf{C}$ is the skew-block diagonal matrix given by,

$$\mathbf{C} = \begin{cases} \mathrm{diag}(c_1\mathbf{J}, \dots, c_{n-1}\mathbf{J}, 0) & \text{if } N = 2n-1, \\ \\ \mathrm{diag}(c_1\mathbf{J}, \dots, c_{n-1}\mathbf{J}, c_n\mathbf{J}) & \text{if } N = 2n, \end{cases} \tag{3.5.7}$$

where $\mathbf{J}$ is as in (3.4.10) and the block entries of $\mathbf{C}$ are given by

$$c_j\mathbf{J} = c_j(k_j)\mathbf{J} = \frac{2\pi a^N b^N}{b^N - a^N} k\mathbf{J}, \tag{3.5.8}$$

Therefore it is not difficult to verify that

$$\Delta u = r^{-1}\left[(N+1)\dot{\mathrm{Q}} + r\ddot{\mathrm{Q}}\right] x = \frac{N^2}{r^{2N+2}}\mathrm{Q}\mathbf{C}^2 x, \tag{3.5.9}$$

and consequentialy (3.5.2) gives that

$$(\nabla u)^t \Delta u = \frac{N^2}{r^{2N+2}}\left(\mathrm{Q}^t + \frac{N}{r^{N+2}} x \otimes \dot{\mathrm{Q}}\mathbf{C}x\right)\mathrm{Q}\mathbf{C}^2 x = \frac{N^2}{r^{2N+2}}\mathbf{C}^2 x. \tag{3.5.10}$$

As the latter must necessarily be *curl-free* (indeed referring to the formulation (3.5.2) the gradient $\nabla\mathfrak{p}$ of the Lagrange multiplier pressure term $\mathfrak{p}$) it thus follows that for each $1 \le i, j \le n$ we have

$$\begin{aligned} \mathbf{curl}\{(\nabla u)^t \Delta u\} = 0 \iff & \frac{\partial}{\partial x_i}\frac{c_j x_j}{r^{2N+2}} - \frac{\partial}{\partial x_j}\frac{c_i x_i}{r^{2N+2}} = 0 \\ \iff & (2N+2)\left(c_j^2 - c_i^2\right)\frac{x_i x_j}{r^{2N+4}} = 0, \end{aligned} \tag{3.5.11}$$

for all $x \in \mathbf{X}$. Thus the whirl mapping u is a solution to the Euler-Lagrange system (3.2.1) *iff* we have $c_j^2 = c_i^2 \equiv c^2$ for all $1 \le i, j \le n$. However in odd dimensions due to the presence of a zero entry in the $\mathbf{C}$ [*cf.* (3.5.7)] this gives $c = 0$ and so $c_i = c_j = 0$ for all $1 \le i, j \le n$ and therefore $\mathrm{Q}^t\dot{\mathrm{Q}} = 0$ that in turn gives

$$\mathrm{Q}^t\dot{\mathrm{Q}} = 0 \iff \dot{\mathrm{Q}} = 0 \iff \mathrm{Q} \equiv \mathrm{I}_N, \tag{3.5.12}$$

as $\mathrm{Q}[a] = \mathrm{Q}[b] = \mathrm{I}_N$. Hence in odd dimensions the only whirl mapping satisfying the Euler-Lagrange system (3.2.1) is the identity mapping $u \equiv x$. In contrast in even dimensions for each $k \in \mathbb{Z}$ we have a solution u to (3.2.1) arising from a whirl function g – given explicitly by (3.4.26) – and with $u(x) = \mathrm{Q}[g](r)x$ where Q is block diagonal (*see* the beginning of the previous section) and $\pm g_1 = \pm g_2 = \cdots = \pm g_n = g$. We have therefore proved the following result.

Theorem 3.5.1. *Let* $\mathbf{X} = \mathbf{X}[a,b] \subset \mathbb{R}^n$ $(n \ge 2)$ *and consider the elastic energy* $\mathbb{E}[u, \mathbf{X}]$ *with* $W(\mathbf{F}) = tr(\mathbf{F}^t\mathbf{F})/2$ *over the space of incompressible admissible mappings* $\mathcal{A}(\mathbf{X})$ *along with the system of Euler-Lagrange equations* (3.2.1). *Then the following hold:*

- *(N even) Here* (3.2.1) *admits infinitely many solutions in the form of whirl mappings, specifically,* $u = \mathrm{Q}[g](r)x$ *with* Q *block diagonal and* $\pm g_1 = \cdots = \pm g_{N/2} = g$ *where the whirl function* g *is given explicitly by* (3.4.26).

- *(N odd) The only solution to* (3.2.1) *in the form of a whirl mapping is the trivial one, that is, the identity mapping* $u \equiv x$.

Remark 3.5.1. In the case that N is even we can explicitly calculate the Dirichlet energy of the whirl solutions as given in Theorem 3.5.1. Namely for each $k \in \mathbb{Z}$ it is seen that the Dirichlet energy of $u = u_k$ is given by

$$\begin{aligned} \mathbb{E}[u; \mathbf{X}] &= \frac{N}{2}|\mathbf{X}| + 2N^2\omega_N \int_a^b \left(\frac{\pi a^N b^N k}{b^N - a^N} \right)^2 r^{-N-1}\, dr \\ &= \frac{N}{2}|\mathbf{X}| + \frac{2N\omega_N \pi^2 a^N b^N}{b^N - a^N} k^2. \end{aligned} \tag{3.5.13}$$

Hence like the earlier planar case the energy of the whirl solutions increase quadratically in k.

Chapter 4

On the Uniqueness of Energy Minimisers in Homotopy Classes

Abstract

In this paper we consider a family of energy functionals $\mathscr{F}$ given by the integral,

$$\mathscr{F}[u;\mathbf{X}] = \int_{\mathbf{X}} \Phi(x,u)|\nabla u|^2 \,\mathrm{d}x,$$

defined over the space of admissible incompressible Sobolev deformations of $\mathbf{X}$,

$$\mathcal{A}(\mathbf{X}) = \Big\{ u \in W^{1,2}(\mathbf{X},\mathbb{R}^2) : \det \nabla u = 1 \ a.e \text{ in } \mathbf{X} \text{ and } u_{|\partial \mathbf{X}} \equiv x \Big\},$$

where $\mathbf{X}$ is the symmetric annular region $\mathbf{X} = \{x \in \mathbb{R}^2 : a < |x| < b\}$ and $\Phi \in \mathbf{C}^1(\mathbb{R}^4)$ is strictly positive. It is well known from [74], [75] or [76] that the space of self mapping $\mathcal{A}(\mathbf{X})$ has the following decomposition, $\mathcal{A}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{A}_k$, where for each $k \in \mathbb{Z}$ the $\mathscr{A}_k$ denotes a homotopy class of $\mathcal{A}(\mathbf{X})$. Through the course of this paper we show that if $0 < \Phi(x,u) = \mathrm{F}(|x|^2, |u|^2)$ where F satisfies certain convexity conditions then each $\mathscr{A}_k$ contains a unique minimisers u_k. Moreover it is shown that these minimisers must be monotone twist mappings, meaning that they are radially symmetric.

Contents

4.1 Introduction

Let $\mathbf{X} \subset \mathbb{R}^2$ denote a smooth bounded domain and consider the class of incompressible deformations,

$$\mathcal{A}(\mathbf{X}) = \{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^2) : \det \nabla u = 1 \text{ } a.e. \text{ in } \mathbf{X}, \ u(x) \equiv x \text{ on } \partial\mathbf{X}\}. \tag{4.1.1}$$

Here the condition of $u(x) \equiv x$ on $\partial\mathbf{X}$ is meant in the sense of traces. Associated to each deformation $u \in \mathcal{A}(\mathbf{X})$ is a corresponding energy given by,

$$\mathscr{F}[u; \mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla u|^2 \, \mathrm{d}x, \tag{4.1.2}$$

where $0 < c \le \mathrm{F} \in \mathbf{C}^\infty(\mathbb{R}^2_+)$. It is known, as a consequence of incompressibility constraint, $u \in W^{1,2}$ and $u \equiv x$ on $\partial\mathbf{X}$, that each $u \in \mathcal{A}(\mathbf{X})$ has a representative belonging to,

$$\mathscr{C}(\mathbf{X}) = \{u \in \mathbf{C}(\overline{\mathbf{X}}, \overline{\mathbf{X}}) : u \equiv x \text{ on } \partial\mathbf{X}\} = \bigcup_{k \in \mathcal{K}} \mathscr{C}_k \tag{4.1.3}$$

Here $\mathcal{K}$ denotes the index set of the family of pairwise disjoint path-connected component (or homotopy classes) of $\mathscr{C}(\mathbf{X})$, whilst $\mathscr{C}_k$ designates the kth homotopy class in the family. Thus,

$$\mathcal{A}(\mathbf{X}) = \bigcup_{k \in \mathcal{K}} \{u \in \mathcal{A}(\mathbf{X}) : u \in \mathscr{C}_k\} = \bigcup_{k \in \mathcal{K}} \mathscr{A}_k, \tag{4.1.4}$$

where above $u \in \mathscr{C}_k$ for $u \in \mathcal{A}(\mathbf{X})$ is meant in terms of the continuous representative. We shall refer to $\mathscr{A}_k$ as the kth homotopy class of $\mathcal{A}(\mathbf{X})$. It can be shown that each of these homotopy classes is sequentially weakly closed with respect to $W^{1,2}$ and therefore $\mathscr{F}[\cdot;\mathbf{X}]$ admits a minimiser u_k in $\mathscr{A}_k$. Furthermore each u_k is a L^1-local minimiser in $\mathcal{A}(\mathbf{X})$ meaning that $\exists\delta = \delta[u_k] > 0$ such that for all $v \in \mathcal{A}(\mathbf{X})$ with $\|u_k - v\|_1 < \delta$ implies that $\mathscr{F}[u_k] \le \mathscr{F}[v]$.

The purpose of this paper is to tackle the question of uniqueness of minimisers to $\mathscr{F}$ in the homotopy classes $\mathscr{A}_k$ for $k \in \mathcal{K}$. To make some traction on this problem we restrict, throughout the paper, to the situation when $\mathbf{X} = \{x \in \mathbb{R}^2 : a < |x| < b\}$ is a two dimensional annulus and here it is known that $\mathcal{K} = \mathbb{Z}$ (*see* [74], [75] or [76]). Through our study we prove in Section 4.5 the following

Main Result: Existence & Uniqueness of Twist Minimisers in $\mathscr{A}_k$

Let $\mathbf{X} = \{x \in \mathbb{R}^2 : a < |x| < b\}$ and consider the energy,

$$\mathscr{F}[u;\mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla u|^2 \, \mathrm{d}x, \tag{4.1.5}$$

over the space of admissble mappings $\mathcal{A}(\mathbf{X})$ while the integrand F lies in,

$$\mathrm{F} \in \mathfrak{F} = \{\mathrm{F} \in \mathbf{C}^\infty(\mathscr{R}) : \mathrm{F} \text{ satisfies } (\mathbf{H1}) \text{ or } (\mathbf{H2})\}$$

where $\mathrm{F} \ge c > 0$, $\mathscr{R} = [a^2, b^2] \times [a^2, b^2]$ and,

H1 $\mathrm{F}(x, y) = \mathrm{F}(1, y)x^{-1}$ and $\partial_y^2[\mathrm{F}(x^2, y^2)^{1/2}] \ge -\partial_y[\mathrm{F}(x^2, y^2)^{1/2}/y]$ for $y \in [a, b]$,

H2 $\mathrm{F}(x, y) = \mathrm{F}(x, 1)y^{-1}$ and $\partial_x^2[\mathrm{F}(x^2, y^2)^{1/2}] \ge -\partial_x[\mathrm{F}(x^2, y^2)^{1/2}/x]$ for $x \in [a, b]$.

Then $\mathscr{F}$ has a unique minimiser u_k in the homotopy class $\mathscr{A}_k$ for each $k \in \mathbb{Z}$. Moreover u_k is a *smooth twist* mapping defined by $u_k(x) = \mathrm{Q}[g_k]x$ where $\mathrm{Q}[\cdot] \in \mathbf{SO}(2)$,

$$g_k(r) = \frac{2\pi k}{\beta}\int_a^r s^{-3}\mathrm{H}(s)^{-1} \, \mathrm{d}s, \tag{4.1.6}$$

with $\mathrm{H}(s) = 2^{-1}\mathrm{F}(r^2, r^2)$ and $\beta = \int_a^b r^{-3}\mathrm{H}(r)^{-1} \, \mathrm{d}r$. Note $g_k \in \mathbf{C}^\infty[a, b]$.

Let us briefly here make a few remarks on this main result. Firstly each mapping $u_k \in \mathscr{A}_k \cap \mathbf{C}^\infty(\overline{\mathbf{X}})$, as is shown in Section 4.4, is a solution to the Euler-Lagrange equation associated to the energy $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$. This Euler-Lagrange equation is given by,

$$\mathbb{EL}[u, \mathfrak{p}; \mathbf{X}] = \begin{cases} \operatorname{div} \mathfrak{S}(x, u, \nabla u) = \partial_\eta \mathrm{F}|\nabla u|^2 u & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u(x) \equiv x & \text{on } \partial\mathbf{X}, \end{cases} \tag{4.1.7}$$

where $\mathrm{F} = \mathrm{F}(|x|^2, |u|^2)$, $\partial_\eta \mathrm{F} = \partial_\eta \mathrm{F}(\mu, \eta)$ and

$$\mathfrak{S}(x, y, \mathbf{F}) = \mathrm{F}(|x|^2, |y|^2)\mathbf{F} - \mathfrak{p}(x)\mathbf{F}^{-t}. \tag{4.1.8}$$

Secondly the mappings $u_k \in \mathscr{A}_k$ being minimisers of F over $\mathscr{A}_k$ for each $k \in \mathbb{Z}$ respectively implies that each u_k is an L^1-local minimiser of F over $\mathcal{A}(\mathbf{X})$, see Section 4.3. Therefore, the main result proves that there are countably many L^1-local minimisers of F over $\mathcal{A}(\mathbf{X})$ in the form of geometrically symmetric twists. Secondly we note that the proof of the main result, in particular when F satisfies (**H1**) relies on the mapping $u \in \mathcal{A}(\mathbf{X})$ being Sobolev homeomorphisms, which is proved in Section 4.2 Theorem 4.2.1.

Let us finish this introduction by highlighting some examples of energies which satisfy the assumptions in the main result above.

- Let $\alpha^2 \geq 1/4$ and $\mathrm{F}(x, y) = x^{2\alpha} y^{-1}$ then the energy,

$$\mathscr{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \frac{|x|^{4\alpha} |\nabla u|^2}{|u|^2} \, \mathrm{d}x, \tag{4.1.9}$$

  satisfies the assumption (**H2**) and note (**H1**). Let us note that when $\alpha = 0$ the twists mappings u_k are minimisers in $\mathscr{A}_k$ for each $k \in \mathbb{Z}$ and is studied by the authors in [58]. However, due to technical reasons our approach here is focused on when F depends on x does not extend to this case when it is independent of x. In this latter case an effective approach is to use the isoperimetric inequality (*see* [58]).

- Let $\alpha^2 \geq 1/4$ and $\mathrm{F}(x, y) = y^{2\alpha} x^{-1}$ then the energy,

$$\mathscr{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \frac{|u|^{4\alpha} |\nabla u|^2}{|x|^2} \, \mathrm{d}x, \tag{4.1.10}$$

  satisfies the assumption (**H1**) and not (**H2**).

- For any $\alpha \geq (\sqrt{2} - 1)/a^2$ and $\mathrm{F}(x, y) = e^{\alpha x} y^{-1}$ then the energy,

$$\mathscr{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \frac{e^{\alpha |x|^2} |\nabla u|^2}{|u|^2} \, \mathrm{d}x, \tag{4.1.11}$$

  satisfies the assumption (**H2**). Note that (**H1**) is not satisfied by this choice of F.

Lastly we observe that the classical Dirichlet energy is not covered by this main theorem as we cannot take $\mathrm{F} \equiv 1$. However, we strongly believe that the uniqueness result holds for that case too but the technical apparatus used here does not cover that case.

4.2 Deformations in $\mathcal{A}(\mathbf{X})$ are Sobolev homeomorphisms

Here we wish to show that each mapping $u \in \mathcal{A}(\mathbf{X})$ is a Sobolev homeomorphism whose inverse mapping $u^{-1} \in \mathcal{A}(\mathbf{X})$. This theorem is essentially a small extension of Theorem 2 from [8]. The main ideas in the proof are from [8], but as the result in our case does not directly follow from there, we give a complete proof here for the ease of the reader. Before continuing we first gather some useful properties of mappings $u \in \mathcal{A}(\mathbf{X})$. Firstly, we recall from the introduction that, any mapping $u \in \mathcal{A}(\mathbf{X})$ has a continuous representative in $\mathbf{C}(\overline{\mathbf{X}}, \mathbb{R}^2)$ which we gain denote by u (*see* Theorem 5.17 in [34]). Additionally, since $u(x) = x$ on $\partial\mathbf{X}$ we know that $\deg(u, \mathbf{X}, p) = \deg(x, \mathbf{X}, p)$ (here $\deg(u, \mathbf{X}, p)$ denotes the Brouwer degree of u at p with respect to $\mathbf{X}$) and moreover $\deg(x, \mathbf{X}, p) = 1$ for all $p \in \mathbf{X}$ and $\deg(x, \mathbf{X}, p) = 0$ for $p \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$. Therefore we obtain that $\overline{\mathbf{X}} \subset u(\overline{\mathbf{X}})$. Furthermore it can been seen that $u(\overline{\mathbf{X}}) \subset \overline{\mathbf{X}}$ by applying Theorem 5.35 in [34]. In particular applying this theorem gives for any $p \in \mathbb{R}^2 \backslash \partial\mathbf{X}$ that,

$$\deg(u, \mathbf{X}, p) = \int_{\mathbf{X}} f(u(x))\, \mathrm{d}x, \tag{4.2.1}$$

when f to continuous real valued function which satisfies $\int_{\mathbb{R}^2} f(x)dx = 1$ and has compact support in the connect component of $\mathbb{R}^2 \backslash \partial\mathbf{X}$ containing p. Now suppose that $\exists x \in \mathbf{X}$ such that $u(x) = p \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$. Then we are free to pick $\delta > 0$ small enough such that $\overline{\mathbb{B}}_\delta(p) \subset \mathbb{R}^2 \backslash \overline{\mathbf{X}}$ and let f have support on $\overline{\mathbb{B}}_\delta(p)$. The continuity of u means that $\exists \xi > 0$ such that $u(\mathbb{B}_\xi(x)) \subset \mathbb{B}_\delta(p)$ and therefore,

$$0 = \deg(u, \mathbf{X}, p) = \int_{\mathbf{X}} f(u(x))\, \mathrm{d}x > 0. \tag{4.2.2}$$

Hence we have reached a contradiction since $\deg(u, \mathbf{X}, p) = 0$ for $p \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$ and thus $\overline{\mathbf{X}} = u(\overline{\mathbf{X}})$. One consequence of this is that $N(u, \mathbf{X}, y) \geq 1$ for $y \in \mathbf{X}$

$$N(y, u, \mathbf{X}) = \sharp\{x \in \mathbf{X} : x \in u^{-1}(y)\} \geq 1 \quad \text{for} \quad y \in \mathbf{X}, \tag{4.2.3}$$

and $N(y, u, \mathbf{X}) = 0$ for $y \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$. Moreover, by Theorem 2.3 on page 285 in [35] we have that,

$$|\mathbf{X}| = \int_{\mathbf{X}} |\det \nabla u|\, \mathrm{d}x = \int_{\mathbb{R}^2} N(y, u, \mathbf{X})\, \mathrm{d}y,$$

which then as $N(y, u, \mathbf{X}) \geq 1$ on $\mathbf{X}$ we conclude,

$$N(y, u, \mathbf{X}) = \sharp\{x \in \mathbf{X} : x \in u^{-1}(y)\} = 1 \quad \text{for } a.e. \ y \in \mathbf{X}. \tag{4.2.4}$$

Hence the mappings $u \in \mathcal{A}(\mathbf{X})$ are injective almost everywhere in $\mathbf{X}$. Furthermore the injectivity *a.e.* of $u \in \mathcal{A}(\mathbf{X})$ gives the following change of variables formula, *see* [35] Theorem 2.4 on page 285,

$$\int_G \varphi \circ u(x)\,\mathrm{d}x = \int_{u(G)} \varphi(y)\,\mathrm{d}y. \tag{4.2.5}$$

For all $G \subset \mathbf{X}$ that are compact domains and $\varphi \in L^1(\mathbb{R}^n)$. Note also that our mapping u satisfies the Luzin N and N^{-1} property, which means the image and pre-image of sets of measure zero under u are also measure zero. This is a consequence of $u \in W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ and $\det \nabla u = 1 > 0$ almost everywhere in $\mathbf{X}$. For a proof of this result the reader is referred to [34] Theorem 5.32 on page 141. We are now in a position to state and prove the aforementioned global invertibility result. Namely,

Theorem 4.2.1. *Let $u \in \mathcal{A}(\mathbf{X})$ then u is a Sobolev homeomorphism with the inverse mapping $u^{-1} \in \mathcal{A}(\mathbf{X})$ and,*

$$\nabla u^{-1}(y) = (\nabla u)^{-1}\,(u^{-1}(y)), \tag{4.2.6}$$

for almost every $y \in \mathbf{X}$.

Proof. We begin by extending the mapping $u \in \mathcal{A}(\mathbf{X})$ by identity onto the ball $\mathbf{Y} = \mathbb{B}_{b+r}$ where $r > 0$ is fixed. Therefore, letting u denote the extended mapping we have that $u \in \mathcal{A}(\mathbf{Y})$. Moreover let $\rho_\varepsilon \geq 0$ be a radial symmetric smooth function with compact support inside $\overline{\mathbb{B}}_\varepsilon(0)$ and with $\int_{\mathbb{R}^2} \rho_\varepsilon(v)\mathrm{d}v = 1$. Now as in [8] pages 320-323 we let for $\varepsilon > 0$,

$$x_\varepsilon(v) = \int_{\mathbf{Y}} \rho_\varepsilon(v - u(y))y\,\mathrm{d}y, \qquad v \in \mathbf{X}, \tag{4.2.7}$$

which will play the role of an approximation of the desired inverse mapping. Then,

$$\begin{aligned} \frac{\partial x_\varepsilon^\alpha}{\partial v^i}(v) &= \int_{\mathbf{Y}} \rho_{\varepsilon,i}(v - u(y))y^\alpha\,\mathrm{d}y \\ &= -\int_{\mathbf{Y}} \frac{\partial \rho_\varepsilon}{\partial y^\beta}(v - u(y))\,(\nabla u)^{-1}_{\beta,i}\,y^\alpha\,\mathrm{d}y. \end{aligned} \tag{4.2.8}$$

Recall that the extended mapping u is uniformly continuous on $\overline{\mathbf{Y}}$ and that we can easily find a sequence of smooth mappings u_s, via mollification, converging strongly in $W^{1,2}(\mathbf{Y}, \mathbb{R}^2)$ and uniformly to u on $\overline{\mathbf{Y}}$. Therefore we can pick $s > N(r)$ such that $\sup_{x\in\overline{\mathbf{X}}}|u_s - u| < r/4$. Additionally as $u = x$ for $x \in \mathbf{Y}\backslash\mathbf{X}$ we know that,

$$\operatorname{dist}(y, \partial\mathbf{Y}) < r/4 \quad \Longrightarrow \quad \operatorname{dist}(u(y), \partial\mathbf{Y}) < r/4. \tag{4.2.9}$$

Thus for any $v \in \mathbf{X}$ we have that,

$$\mathbb{B}_\varepsilon(v) \cap \mathbb{B}_{r/4}(u_s(y)) = \varnothing, \tag{4.2.10}$$

for all $y \in \mathbf{Y}$ such that $\operatorname{dist}(y, \partial \mathbf{X}) < r/4$ and $s > N(r/4)$ provided that $\varepsilon < r/4$. Hence,

$$\rho_\varepsilon(v - u_s(y)) = 0, \qquad \forall v \in \mathbf{X}. \tag{4.2.11}$$

Then by using the divergence free structure of adj, which is the transpose of the cofactor matrix, it follows via an application of divergence theorem that,

$$-\int_{\mathbf{Y}} \frac{\partial \rho_\varepsilon}{\partial y^\beta}(v - u_s(y))\,(\mathrm{adj}\nabla u_s)_{\beta,i}\, y^\alpha \,\mathrm{d}y = \int_{\mathbf{Y}} \rho_\varepsilon(v - u_s(y))\,(\mathrm{adj}\nabla u_s)_{\alpha,i}\,\mathrm{d}y \tag{4.2.12}$$

for all $v \in \mathbf{X}$ provided $s > N(r/4)$. Then since $\det \nabla u = 1$ *a.e.* and $u_s \to u$ in $W^{1,2}(\mathbf{Y}, \mathbb{R}^2)$ we obtian that $\mathrm{adj}\nabla u_s \to (\nabla u)^{-1}$ in $L^2(\mathbf{Y})$. Note here that we have used that $\mathrm{adj}\nabla u = (\nabla u)^{-1}$. Thus by passing to the limit in the above we obtain that,

$$\forall v \in \mathbf{X}, \qquad \frac{\partial x_\varepsilon^\alpha}{\partial v^i}(v) = \int_{\mathbf{Y}} \rho_\varepsilon(v - u(y))\,(\nabla u)^{-1}_{\alpha,i}\,\mathrm{d}y, \tag{4.2.13}$$

provided that $\varepsilon < r/4$. Also note that $\|x_\varepsilon\|_{L^2} \leq C$ and by the above,

$$\begin{aligned} |\nabla x_\varepsilon|^2 &\leq \left(\int_{\mathbf{Y}} \rho_\varepsilon(v - u(y))|(\nabla u)^{-1}|\,\mathrm{d}y\right)^2 \\ &\leq \left(\int_{\mathbf{Y}} \rho_\varepsilon(v - u(y))\,\mathrm{d}y\right)\left(\int_{\mathbf{Y}} \rho_\varepsilon(v - u(y))|(\nabla u)^{-1}|^2\,\mathrm{d}y\right). \end{aligned}$$

Moreover, we know $|(\nabla u)^{-1}|(x) = |\nabla u|(x)$ for *a.e.* $x \in \mathbf{Y}$ as we are in two dimensions and $\det \nabla u = 1$ *a.e.* Additionally for $v \in \mathbf{X}$ with $\varepsilon < r/4$ we have that,

$$|\nabla x_\varepsilon(v)|^2 \leq \int_{\mathbf{Y}} \rho_\varepsilon(v - u(y))|\nabla u|^2\,\mathrm{d}y. \tag{4.2.14}$$

Therefore integrating both sides and applying Fubini theorem gives that,

$$\int_{\mathbf{X}} |\nabla x_\varepsilon(v)|^2\,\mathrm{d}v \leq \int_{\mathbf{Y}} |\nabla u|^2 \mathrm{d}y. \tag{4.2.15}$$

Thus $\{x_\varepsilon\}$ is a bounded sequence in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ provided that $\varepsilon < r/4$ and hence the sequence $\{x_\varepsilon\}_{\varepsilon < r/4}$ has a subsequence, not re-labelling, where $x_\varepsilon \rightharpoonup x$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$. Therefore for any compact domain $A \subset \mathbf{X}$ we know that,

$$\int_A \frac{\partial x_\varepsilon^\alpha}{\partial v^i}(v)\,\mathrm{d}v \to \int_A \frac{\partial x^\alpha}{\partial v^i}(v)\,\mathrm{d}v. \tag{4.2.16}$$

To utilise this we integrate (4.2.13) over A and apply Fubini theorem to obtain that,

$$\begin{aligned} \int_A \frac{\partial x_\varepsilon^\alpha}{\partial v^i}(v)\,\mathrm{d}v &= \int_{\mathbf{Y}} \int_A \rho_\varepsilon(v - u(y))\mathrm{d}v\,(\nabla u)^{-1}_{\alpha,i}\,\mathrm{d}y \\ &= \int_{\mathbf{Y}} \rho_\varepsilon * \chi_A(u(y))\,(\nabla u)^{-1}_{\alpha,i}\,\mathrm{d}y. \end{aligned} \tag{4.2.17}$$

Thus combining this with (4.2.16) gives the following identity,

$$\int_{\mathbf{Y}} (\nabla u)^{-1}{}_{i,\alpha}(y)\chi_A(u(y))\,\mathrm{d}y = \int_A \frac{\partial x^\alpha}{\partial v^i}(v)\,\mathrm{d}v \tag{4.2.18}$$

Let us now pick $A = u(\overline{\mathbb{B}}_s(\zeta))$ where $\zeta \in \mathbf{X}$ and $s > 0$ is such that $\mathbb{B}_s(\zeta) \subset \mathbf{X}$. Then as u is injective *a.e.* and satisfies the Luzin N^{-1} property we know that $u^{-1}(A) = \overline{\mathbb{B}}_s(\zeta)$ up to a set of measure zero. Thus,

$$\int_{\mathbb{B}_s(\zeta)} (\nabla u)^{-1}{}_{i,\alpha}(y)\,\mathrm{d}y = \int_{u(\overline{\mathbb{B}}_s(\zeta))} \frac{\partial x^\alpha}{\partial v^i}(v)\,\mathrm{d}v = \int_{\mathbb{B}_s(\zeta)} \frac{\partial x^\alpha}{\partial v^i}(u(y))\,\mathrm{d}y, \tag{4.2.19}$$

The last equality here comes from (4.2.5) and noting that $G = \overline{\mathbb{B}}_s(\zeta)$. Hence,

$$\int_{\mathbb{B}_s(\zeta)} (\nabla u)^{-1}{}_{i,\alpha}(y)\,\mathrm{d}y = \int_{\mathbb{B}_s(\zeta)} \frac{\partial x^\alpha}{\partial v^i}(u(y))\,\mathrm{d}y, \tag{4.2.20}$$

for all $s < \mathrm{dist}(\zeta, \partial\mathbf{X})$ and any $\zeta \in \mathbf{X}$. Then by an application of the Lebesgue differentiation theorem we have that,

$$(\nabla u)^{-1}{}_{i,\alpha}(y) = \frac{\partial x^\alpha}{\partial v^i}(u(y)), \tag{4.2.21}$$

for *a.e.* $y \in \mathbf{X}$ and hence $(\nabla u)^{-1}(y) = \nabla x(u(y))$ for *a.e.* $y \in \mathbf{X}$. Now let,

$$\mathcal{S} = \{y \in \mathbf{X} : (4.2.21)\text{ fails}\}, \tag{4.2.22}$$

then by the Luzin N property we know that $u(\mathcal{S}) \subset \mathbf{X}$ has measure zero. Hence combining this with the almost everywhere injectivity of u implies that,

$$(\nabla u)^{-1}{}_{i,\alpha}(u^{-1}(y)) = \frac{\partial x^\alpha}{\partial v^i}(y), \tag{4.2.23}$$

for *a.e.* $y \in \mathbf{X}$. Furthermore,

$$\det \nabla x(y) = \det (\nabla u)^{-1}(u^{-1}(y)) = \det \nabla u(u^{-1}(y)) = 1,$$

almost everywhere in $\mathbf{X}$ which gives that $x(\cdot)$ is continuous.

Recall that $x_\varepsilon \rightharpoonup x$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ then on a subsequence, again not re-labelling, $x_\varepsilon \to x$ for almost every $v \in \mathbf{X}$. Additionally since $u(\cdot)$ is injective almost everywhere we let $N \subset \mathbf{X}$ denote the set of points where u fails to be injective or x_ε fails to converge pointwise. Then for $v \in \mathbf{X}\backslash N$ we have that $\exists! z \in \mathbf{X}$ s.t $u(z) = v$ and,

$$x_\varepsilon(v) - z = \int_{\mathbf{Y}} \rho_\varepsilon(u(z) - u(y))(y - z)\,\mathrm{d}y. \tag{4.2.24}$$

Then by the uniqueness of z and the uniform continuity of u on $\overline{\mathbf{X}}$ we have that $\forall \mu > 0, \exists \delta(\mu)$ such that $|z-y| < \mu$ if $|u(z)-u(y)| \leq \delta$. Take $\varepsilon < \delta(\mu)$ then,

$$|x_\varepsilon(v) - z| \leq \mu \int_{\mathbf{Y}} \rho_\varepsilon(u(z) - u(y))\, \mathrm{d}y = \mu, \tag{4.2.25}$$

which therefore gives that $x_\varepsilon(v) \to x(v) = z$ as $\varepsilon \to 0$ for all $v \in \mathbf{X}\backslash N$. Hence $u(x(v)) = u(z) = v$ for all $v \in \mathbf{X}\backslash N$ and therefore by the continuity of $x(\cdot)$ and $u(\cdot)$ we have that $u(x(v)) = v$ for all $v \in \mathbf{X}$.

To see that it is a left inverse let $y \in \mathbf{X}$ and take $\{y_n : n \in \mathbb{N}\} \subset \mathbf{X}$ such that $y_n \to y$ with $z_n = u(y_n) \in \mathbf{X}\backslash N$. Then we have that $u(v) = z_n$ if and only if $v = y_n$ and therefore using this with $u(x(v)) = v$ for all $v \in \mathbf{X}$ we conclude that,

$$x(u(y_n)) = y_n \implies x(u(y)) = y, \tag{4.2.26}$$

by the continuity of both $x(\cdot)$ and $u(\cdot)$. Thus $x(u(y)) = y$ for all $y \in \mathbf{X}$. One immediate consequence of this is that (4.2.23) gives,

$$\nabla x(v) = (\nabla u)^{-1}(x(v)), \tag{4.2.27}$$

for almost every $v \in \mathbf{X}$. Additionally if we let $y \in \partial\mathbf{X}$ and take $\{y_n\} \subset \mathbf{X}$ such that $y_n \to y$ then as,

$$\begin{aligned}
\lim_{n\to\infty} u(y_n) &= u(y) = y, \\
\lim_{n\to\infty} x(u(y_n)) &= x(u(y)), \\
\lim_{n\to\infty} x(u(y_n)) &= \lim_{n\to\infty} y_n = y.
\end{aligned}$$

Thus, $x(y) = y$ on $\partial\mathbf{X}$ and $x(\cdot)$ is the global inverse of $u \in \mathcal{A}(\mathbf{X})$ and moreover $x = u^{-1} \in \mathcal{A}(\mathbf{X})$. □

Remark 4.2.1. Let us finish this section by outlining an alternative proof to Theorem 4.2.1. Given any $u \in \mathcal{A}(\mathbf{X})$ we begin by extending u by identity onto the ball $\mathbb{B}_{2R}$ where $R > b$, i.e. $\overline{\mathbf{X}} \subset \mathbb{B}_R$, and therefore $u \in \mathcal{A}(\mathbb{B}_{2R})$. Now by Theorem 1 in [50] there is a Stoilow's type factorisation of $u \in \mathcal{A}(\mathbb{B}_{2R})$. Namely there exists a homeomorphism $h \in W^{1,2}(\Omega, \mathbb{B}_{2R})$ and a holomorphic mapping $\varphi \in (\Omega, \mathbb{C})$, where $\Omega \subset \mathbb{C}$ is some open set, such that,

$$u = \varphi \circ h^{-1}. \tag{4.2.28}$$

Furthermore, as $h^{-1}(\overline{\mathbb{B}}_R) \subset \Omega$ is a compact subset we know, since points where the derivative of φ vanishes are isolated, that φ restricted to $\overline{\mathbb{B}}_R$ is locally conformal at all but finitely many points. Thus, as h is a homeomorphism, u is a local homeomorphism at all but finitely many points on $\mathbb{B}_R$. Moreover (*see* [56]) since $\partial\mathbb{B}_R$ is connected, u is one-to-one on $\partial\mathbb{B}_R$ and a local homeomorphism at all but finitely many points we obtain that u is a global homeomorphism on $\mathbb{B}_R$, which in turn implies that u is a global homeomorphism on $\overline{\mathbf{X}}$.

However we note that the proof presented above is more direct and avoids Stoilow's factorisation and the machinery complex function theory.

4.3 The homotopic structure of $\mathcal{A}(\mathbf{X})$ and the corresponding existence of countably many L^1-local minimisers

The purpose of this section is to pull together some results from [74]-[76], which were eluded to in the introduction, regarding the homotopic structure of $\mathcal{A}(\mathbf{X})$. These results provide the existence of minimisers to $\mathscr{F}$ in the homotopy class $\mathscr{A}_k$ for each $k \in \mathbb{Z}$. Each of these minimisers are then shown to be L^1 local minimisers of $\mathscr{F}$ in $\mathcal{A}(\mathbf{X})$.

Definition 4.3.1. Let $\mathbf{X} = \mathbf{X}[a,b] = \{x \in \mathbb{R}^2 : a < |x| < b\}$ with $0 < a < b < \infty$. Then we set

$$\mathscr{C} = \mathscr{C}(\mathbf{X}) := \left\{ f \in \mathbf{C}(\overline{\mathbf{X}}, \overline{\mathbf{X}}) : f(x) \equiv x \text{ on } x \in \partial\mathbf{X} \right\}. \tag{4.3.1}$$

We equip $\mathscr{C}(\mathbf{X})$ with the topology of uniform convergence.

Interest in this space of continuous self-mapping comes from the fact that the space $\mathcal{A}(\mathbf{X})$ can be 'embedded' into $\mathscr{C}(\mathbf{X})$. Note that in Section 4.2 we proved that each $u \in \mathcal{A}(\mathbf{X})$ had a representative which was a homeomorphism. Namely each $u \in \mathcal{A}(\mathbf{X})$ has a homeomorphism representation which again is denoted by u in

$$\mathscr{H}(\mathbf{X}) := \{u \in \mathscr{C}(\mathbf{X}) : u \text{ is a homeomorphism}\} \subset \mathscr{C}(\mathbf{X}). \tag{4.3.2}$$

Furthermore associated to each $u \in \mathscr{H}(\mathbf{X})$ is the following topological invariant $\mathbf{deg}(u)$ which denotes the winding number of the planar curve $\gamma_\theta(r) = u/|u|(r,\theta) : [a,b] \to \mathbb{S}^1$ and $\theta \in [0, 2\pi)$ is fixed. Note that by the continuity of u and the winding number being integer valued this is independent of the particular choice of $\theta \in [0, 2\pi)$. Moreover the $\mathbf{deg}(u)$ induces an enumeration of the homotopy classes of $\mathscr{H}(\mathbf{X})$ which is summarised by the following proposition.

Proposition 4.3.1. *Let* $\mathscr{H}_k = \{u \in \mathscr{H}(\mathbf{X}) : \mathbf{deg}(u) = k\}$ *for* $k \in \mathbb{Z}$. *Then* $\mathscr{H}_k$ *are pairwise disjoint and,*

$$\mathscr{H}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{H}_k. \tag{4.3.3}$$

Furthermore the degree map $\mathbf{deg} : \{[u] : u \in \mathscr{H}\} \to \mathbb{Z}$ *is a bijection.*

Note that if we assume some further differentiability on the mapping $u \in \mathscr{H}$ then we have the following integral formula:

$$\mathbf{deg}(u) = \frac{1}{2\pi}\int_a^b \frac{u \times \partial_r u}{|u|^2}(r\omega)\,\mathrm{d}r, \tag{4.3.4}$$

where $\omega = x/|x| \in \mathbb{S}^1$ and $r = |x|$. Let us briefly remark here that $\mathbf{deg}(u)$ should not be confused with the Brouwer degree. In this context every mappings $u \in \mathcal{A}(\mathbf{X})$ would have Brouwer degree 1. The degree formula (4.3.4) identifies the homotopy class membership of $u \in \mathscr{C}(\mathbf{X})$ in (4.1.3). Now utilising that each mapping $u \in \mathcal{A}(\mathbf{X})$ has a homeomorphism representative (Theorem 4.2.1) we find that it is possible to write $\mathcal{A}(\mathbf{X})$ in the following way,

$$\mathcal{A}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{A}_k \quad \text{where} \quad \mathscr{A}_k = \Big\{u \in \mathcal{A}(\mathbf{X}) : \mathbf{deg}\,(u) = k\Big\}, \tag{4.3.5}$$

are pairwise disjoint. Above we have made a slight abuse of notation as $\mathbf{deg}(u)$ is meant as described above for the homeomorphic representative of $u \in \mathcal{A}(\mathbf{X})$. The main purpose of this homotopic structure for us is that we gain the existence of countable many L^1 local minimisers of $\mathscr{F}$ via the following result.

Theorem 4.3.2. *Let* $k \in \mathbb{Z}$ *then there exists* $u_k = u(x;k) \in \mathscr{A}_k$ *such that*

$$\mathscr{F}[u_k;\mathbf{X}] = \inf_{v\in\mathscr{A}_k} \mathscr{F}[v;\mathbf{X}]. \tag{4.3.6}$$

Furthermore for each such minimiser u *there exists* $\delta = \delta(u) > 0$ *such that*

$$\mathscr{F}[u_k;\mathbf{X}] \le \mathscr{F}[v;\mathbf{X}], \tag{4.3.7}$$

for all $v \in \mathcal{A}(\mathbf{X})$ *satisfying* $||u_k - v||_{L^1} < \delta$. *Thus* u_k *is a local minimiser of* $\mathscr{F}$ *in* $\mathcal{A}(\mathbf{X})$ *in the* L^1*-metric.*

Proof. Firstly fix $k \in \mathbb{Z}$ and pick $(u_j) \subset \mathscr{A}_k$ to be an *infimizing* sequence, i.e. $\mathscr{F}[u_j] \downarrow \alpha := \inf_{\mathscr{A}_k} \mathscr{F}[\cdot]$. Furthermore as $a \le |x|, |u| \le b$ with $\alpha < \infty$ we can pass to a subsequence (not re-labeled) $u_j \rightharpoonup u$ in $W^{1,2}(\mathbf{X},\mathbb{R}^2)$. Moreover it can be shown (*see* the proof of Proposition 4.3 on page 404 in [74]) that we can extract a further subsequence (u_j) such that $u_j \to u$

uniformly in $\overline{\mathbf{X}}$ which in turn implies that $u \in \mathscr{A}_k$. Now to obtain existence of minimisers in $\mathscr{A}_k$ we just need to prove that $\mathscr{F}$ is sequentially weakly lower semicontinuous. To this end note that,

$$
\begin{aligned}
\left| \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u_j|^2)|\nabla u_j|^2 - \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla u_j|^2 \right| & \\
&\leq \int_{\mathbf{X}} |\nabla u_j|^2 \left| \mathrm{F}(|x|^2, |u_j|^2) - \mathrm{F}(|x|^2, |u|^2) \right| \\
&\leq \sup_{\overline{\mathbf{X}}} \left| \mathrm{F}(|x|^2, |u_j|^2) - \mathrm{F}(|x|^2, |u|^2) \right| \int_{\mathbf{X}} |\nabla u_j|^2 \to 0
\end{aligned}
$$

as $j \nearrow \infty$ as a result of the $W^{1,2}$ boundedness of $\{u_j\}$ and the uniform convergence $u_j \to u$ on $\overline{\mathbf{X}}$. Combining this with

$$
\int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla u|^2 \leq \underline{\lim} \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla u_j|^2 \tag{4.3.8}
$$

gives the desired lower semicontinuity of the $\mathscr{F}$ energy on $\mathcal{A}(\mathbf{X})$ as claimed. To justify the L^1-local minimiser claim we can argue by contradiction. Suppose that u_k is not a L^1-local minimiser; then $\exists (v_n : n \in \mathbb{N}) \subset \mathcal{A}(\mathbf{X})$ such that $||v_n - u_k||_{L^1} \to 0$ as $n \nearrow \infty$ and moreover $\mathscr{F}[v_n] < \mathscr{F}[u_k]$ for every $n \in \mathbb{N}$. Now recall that $0 < c \leq \mathrm{F}$ and therefore,

$$
c\|\nabla v_n\|_2^2 < \mathscr{F}[u_k], \tag{4.3.9}
$$

which in turn gives that (v_n) is bounded in $W^{1,2}$. Thus as earlier it is possible to extract a subsequence such that v_n converges to u_k weakly in $W^{1,2}$ and uniformly in $\overline{\mathbf{X}}$. Hence as $\mathbf{deg}(v_n)$ is integer valued $\exists N > 0$ such that $v_n \in \mathscr{A}_k(\mathbf{X})$ for $n \geq N$. This is a contradiction as $\mathscr{F}[v_n] < \mathscr{F}[u_k] = \inf_{u \in \mathscr{A}_k(\mathbf{X})} \mathscr{F}$. Hence u_k must be an L^1-local minimiser as claimed. □

The main aim of this paper is to strengthen this result to the existence of countably many L^1 local minimisers, in the form of geometrically symmetric twists, by proving that the minimiser in each $\mathscr{A}_k$ is a unique twist mapping. The precise definition of a twist mapping and their properties will be given in the next subsection.

4.4 Twist mappings and the existence of countably many stationary twists

The aim of this section is to prove that the existence of countably many critical points to F in the form of a twist mapping. In Section 4.5 we shall in fact see that each of these twist critical points are unique minimisers in their respective homotopy class $\mathscr{A}_k$.

Before beginning the task of proving this result let us first take a moment to define twist mappings and prove some of their properties.

Definition 4.4.1. A mapping $u \in \mathscr{C}(\mathbf{X})$ is a called a twist mapping if it has the following form,

$$u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} \cos g(r) & -\sin g(r) \\ \sin g(r) & \cos g(r) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{4.4.1}$$

for some function $g \in \mathbf{C}[a,b]$ and $r = \sqrt{x_1^2 + x_2^2}$. For ease we shall write $u(x) = \mathrm{Q}[g](r)x$ where $\mathrm{Q}[\cdot] \in \mathbf{SO}(2)$ is clear from the above.

Note that if we identify $x = (x_1, x_2)$ with $z = x_1 + ix_2 \in \mathbb{C}$ and $u = (u_1, u_2)$ with $w(z) = u_1(x_1, x_2) + iu_2(x_1, x_2)$ then the representation (4.4.1) is equivalent to,

$$w(z) = e^{ig(r)}z. \tag{4.4.2}$$

However (4.4.1) is used as it leads itself more immediately to generalisations to higher dimensions (*see* [64], [75], [76] and [77]).

Furthermore it can be easily seen, with some added assumption on the differentiability of g, that the corresponding twist mapping $u \in \mathcal{A}(\mathbf{X})$. Namely suppose that $g(a) = 2\pi k_1$ and $g(b) = 2\pi k_2$ for some $k_1, k_2 \in \mathbb{Z}$, whilst being suitably differentiable. Then we have that $u(x) = x$ on $\partial\mathbf{X}$ and,

$$\nabla u = \mathrm{Q}[g] + r\dot{g}(r)\dot{\mathrm{Q}}[g]\theta \otimes \theta \implies |\nabla u|^2 = 2 + r^2|\dot{g}|^2. \tag{4.4.3}$$

Furthermore it can also be seen that

$$\det \nabla u = \det\Big(\mathrm{Q}[g] + r\dot{g}(r)\dot{\mathrm{Q}}[g]\theta \otimes \theta\Big) = 1, \tag{4.4.4}$$

and therefore $u \in \mathcal{A}(\mathbf{X})$ provided that,

$$g \in \mathscr{G} = \{g \in W^{1,2}[a,b] : g(a) = 2\pi k_1, \;\; g(b) = 2\pi k_2 \text{ where } k_1, k_2 \in \mathbb{Z}\}. \tag{4.4.5}$$

Moreover for twist mapping it is also straightforward to see that,

$$\mathbf{deg}(u) = \frac{1}{2\pi}\int_a^b \frac{u \times \partial_r u}{|u|^2}(r\omega)\,\mathrm{d}r = \frac{1}{2\pi}\int_a^b \dot{g}(r)\,\mathrm{d}r = k_2 - k_1. \tag{4.4.6}$$

Hence the above calculations have proven the following,

Proposition 4.4.1. *For each $g \in \mathscr{G}$ the corresponding twist mapping u given by Definition 4.4.1 lies in $\mathscr{A}_{k_2 - k_1}$.*

Remark 4.4.1. By the definition of a twist mapping (4.4.1) clearly defines a mapping that is invariant under $\mathbf{SO}(2)$ in the sense that,

$$u(x) = R^t u(Rx), \qquad \forall R \in \mathbf{SO}(2). \tag{4.4.7}$$

Thus the twist mapping posses a large amount of symmetry in $\mathbb{R}^2$ and are radially symmetric.

We now wish to prove that there are countably many twist mappings which are stationary points to the energy $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$. To this end we note the Euler-Lagrange system associated to F over $\mathcal{A}(\mathbf{X})$ is,

$$\mathbb{EL}[u, \mathfrak{p}; \mathbf{X}] = \begin{cases} \operatorname{div} \mathfrak{S}(x, u, \nabla u) = \partial_\eta \mathrm{F} |\nabla u|^2 u & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u(x) \equiv x & \text{on } \partial \mathbf{X}, \end{cases} \tag{4.4.8}$$

where $\mathrm{F} = \mathrm{F}(|x|^2, |u|^2)$, $\partial_\eta \mathrm{F} = \partial_\eta \mathrm{F}(\mu, \eta)$ and

$$\mathfrak{S}(x, u, \nabla u) = \mathrm{F}(|x|^2, |u|^2) \nabla u - \mathfrak{p}(x)(\nabla u)^{-t}. \tag{4.4.9}$$

Now we recall that a mapping $u \in \mathcal{A}(\mathbf{X})$ is a stationary point of $\mathscr{F}$ if there exists some $\mathfrak{p}$ such that the pair $(u, \mathfrak{p})$ are a classical solution to the Euler-Lagrange system. Meaning that $u \in \mathbf{C}^2(\mathbf{X}, \mathbb{R}^2) \cap \mathbf{C}(\overline{\mathbf{X}}, \mathbb{R}^2)$ and $\mathfrak{p} \in \mathbf{C}^1(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$ and $\mathbb{EL}[u, \mathfrak{p}; \mathbf{X}]$ is satisfied. In order to achieve our aim set for this section we begin by restricting our energy to the class of twist mapping in $\mathcal{A}(\mathbf{X})$. This reduced energy has a corresponding restricted Euler-Lagrange which will turn out to be an ODE. Solutions to this restricted Euler-Lagrange give twist mappings which are critical points of the reduced energy. These are then our candidates for twist mappings which are critical points of the full energy, i.e. solutions to (4.4.8). Firstly $\mathscr{F}$ over twist is,

$$\begin{aligned} \mathscr{F}[u; \mathbf{X}] &= \int_{\mathbf{X}} 2^{-1} \mathrm{F}(r^2, r^2) |\mathrm{Q}(g(r)) + r \dot{g}(r) \dot{\mathrm{Q}} \theta \otimes \theta|^2 \, \mathrm{d}x \\ &= \int_{\mathbf{X}} \mathrm{H}(r) \left(2 + 2r\dot{g} \langle \mathrm{Q}^t \dot{\mathrm{Q}} \theta, \theta \rangle + r^2 \dot{g}^2 |\mathrm{Q}^t \dot{\mathrm{Q}} \theta| \right) \, \mathrm{d}x \\ &= 2\pi \int_a^b \mathrm{H}(r) \left(2 + r^2 \dot{g}^2 \right) \, r \mathrm{d}r = C_1 + 2\pi \mathscr{W}[g; a, b]. \end{aligned} \tag{4.4.10}$$

Above we have let $0 < c \leq \mathrm{H}(r) = \frac{1}{2} \mathrm{F}(r^2, r^2)$, $C_1 = 2\pi \int_a^b r \mathrm{H}(r) \, \mathrm{d}r$ and defined the energy $\mathscr{W}[g; a, b]$ by,

$$\mathscr{W}[g; a, b] = \int_a^b r^3 \mathrm{H}(r) \dot{g}^2 \, \mathrm{d}r. \tag{4.4.11}$$

It is then straightforward to show that the restricted Euler-Lagrange of the energy $\mathscr{F}$ for twist mapping in $\mathcal{A}(\mathbf{X})$ is given by,

$$\frac{\partial}{\partial r}\left(r^3\mathrm{H}(r)\dot{g}\right)=0,\ \ g(a)=2\pi k_1,\quad g(b)=2\pi k_2. \tag{4.4.12}$$

Provided that $\mathrm{H}(r)>0$ solutions to (4.4.12) are given by,

$$g(r)=c_1\int_a^r\frac{\mathrm{d}s}{s^3\mathrm{H}(s)}+c_2, \tag{4.4.13}$$

where $c_2=2\pi k_1$ and $c_1=2\pi(k_2-k_1)/\beta$ with $\beta=\int_a^b r^{-3}\mathrm{H}(r)^{-1}\mathrm{d}r$. Hence without loss of generality we can assume that $k_1=0$ with $k_2=k$ and therefore for each $k\in\mathbb{Z}$ we have that there exists a g_k given by,

$$g_k(r)=\frac{2\pi k}{\beta}\int_a^r\frac{\mathrm{d}s}{s^3\mathrm{H}(s)}, \tag{4.4.14}$$

where $\beta=\int_a^b r^{-3}\mathrm{H}(r)^{-1}\,\mathrm{d}r$. We shall from now on denote $u_k\in\mathscr{A}_k$ to be the twist mapping that corresponds to the *angle of rotation* function g_k given by (4.4.14).

We now aim to prove that our candidate twist mapping associated to the *angle of rotation* function g_k, given by (4.4.14), are critical points to the full Euler-Lagrange (4.4.8). Namely,

Theorem 4.4.2. *Let $\mathbf{X}=\mathbf{X}[a,b]\subset\mathbb{R}^2$ and consider the $\mathscr{F}$ energy given by,*

$$\mathscr{F}[u;\mathbf{X}]=\frac{1}{2}\int_{\mathbf{X}}\mathrm{F}(|x|^2,|u|^2)|\nabla u|^2\,\mathrm{d}x, \tag{4.4.15}$$

over the space of admissible mappings $\mathcal{A}(\mathbf{X})$. Then the twist mappings given by $u_k=\mathrm{Q}[g_k]x$, where g_k is given by (4.4.14) *for each $k\in\mathbb{Z}$, are solutions of the Euler-Lagrange equation* (4.4.8).

Proof. Firstly through direct computation we can show that for every twist mapping the following identities hold:

$$\nabla u=\mathrm{Q}+r\dot{g}\dot{\mathrm{Q}}\theta\otimes\theta,\quad |\nabla u|^2=2+r^2\dot{g}^2,\quad \Delta u=\left[3\dot{g}\dot{\mathrm{Q}}+r\ddot{g}\dot{\mathrm{Q}}-r\dot{g}^2\mathrm{Q}\right]\theta.$$

Again straightforward computations show that,

$$\begin{aligned}\mathbf{I}_1&=(\nabla u)^t\nabla u(\nabla u)^t u=(1+r^2\dot{g}^2)x+r\dot{g}\mathrm{J}x \qquad (4.4.16)\\ &=(\nabla u)^t(\nabla u)x,\\ \mathbf{I}_2&=|\nabla u|^2(\nabla u)^t u=(2+r^2\dot{g}^2)x, \qquad (4.4.17)\\ \mathbf{I}_3&=(\nabla u)^t\Delta u=\left[\left(3r^{-1}\dot{g}+r\ddot{g}\right)\mathrm{J}+2\dot{g}^2\mathrm{I}\right]x. \qquad (4.4.18)\end{aligned}$$

Then we can write the Euler-Lagrange equation (4.4.8) in the following way,

$$\nabla \mathfrak{p} = 2\left(\partial_\xi \mathrm{F} + \partial_\eta \mathrm{F}\right)\mathbf{I}_1 - \partial_\eta \mathrm{F}\mathbf{I}_2 + \mathrm{F}\mathbf{I}_3. \tag{4.4.19}$$

Note that $\mathrm{F} = \mathrm{F}(|x|^2, |x|^2) = 2\mathrm{H}(r)$ and therefore $\dot{\mathrm{H}}(r)/r = \partial_1 \mathrm{F} + \partial_2 \mathrm{F}$. Thus,

$$\nabla \mathfrak{p} = 2r^{-1}\dot{\mathrm{H}}\mathbf{I}_1 - \partial_\eta \mathrm{F}\mathbf{I}_2 + 2\mathrm{H}\mathbf{I}_3. \tag{4.4.20}$$

Assuming that g satisfies (4.4.12) then $\mathrm{H}\mathbf{I}_3$ reduces to the following,

$$\mathrm{H}\mathbf{I}_3 = r^{-3}\left[3r^2\mathrm{H}\dot{g} + r^3\ddot{g}\mathrm{H}\right]\mathrm{J}x + 2\dot{g}^2\mathrm{H}x = -\dot{g}\dot{\mathrm{H}}\mathrm{J}x + 2\dot{g}^2\mathrm{H}x. \tag{4.4.21}$$

Moreover,

$$r^{-1}\dot{\mathrm{H}}\mathbf{I}_1 + \mathrm{H}\mathbf{I}_3 = r^{-1}\dot{\mathrm{H}}\left(1 + r^2\dot{g}^2\right)x + 2\dot{g}^2\mathrm{H}x. \tag{4.4.22}$$

Therefore substituting (4.4.22) into (4.4.20) gives that,

$$\begin{aligned}\nabla \mathfrak{p} &= 2r^{-1}\dot{\mathrm{H}}\left(1 + r^2\dot{g}^2\right)x + 4\dot{g}^2\mathrm{H}x - \partial_\eta \mathrm{F}\mathbf{I}_2 \\ &= 4c^2r^{-6}\mathrm{H}^{-1}x + 2r^{-1}\dot{\mathrm{H}}\left(1 + c^2r^{-4}\mathrm{H}^{-2}\right)x - \partial_\eta \mathrm{F}\left(2 + c^2r^{-4}\mathrm{H}^{-2}\right)x.\end{aligned} \tag{4.4.23}$$

Above we have used the form of g, namely (4.4.14), and let $c = 2\pi k {C_2}^{-1}$. Now since,

$$\nabla \mathrm{H}(r) = r^{-1}\dot{\mathrm{H}}x, \tag{4.4.24}$$

we are left with to that $4c^2r^{-6}\mathrm{H}^{-1}x$, $2c^2r^{-5}\dot{\mathrm{H}}\mathrm{H}^{-2}x$ and $\partial_\eta \mathrm{F}\left(2 + c^2r^{-4}\mathrm{H}^{-2}\right)x$ can be written as a gradient of some functions in order for u_k to be a solution to (4.4.8). This can be easily done by letting,

$$G_1(r) = 2\int_a^r c^2 s^{-4}\mathrm{H}^{-2}\dot{\mathrm{H}}\,\mathrm{d}s, \tag{4.4.25}$$

and noting that $\nabla G_1(r) = 2c^2r^{-5}\dot{\mathrm{H}}\mathrm{H}^{-2}x$. In a similar fashion we let,

$$G_2(r) = \int_a^r s\,\partial_\eta \mathrm{F}\left(2 + c^2s^{-4}\mathrm{H}^{-2}\right)\mathrm{d}s, \tag{4.4.26}$$

which then satisfies $\nabla G_2(r) = \partial_2 \mathrm{F}(r^2, r^2)\left(2 + c^2r^{-4}\mathrm{H}^{-2}\right)$. Finally let,

$$G_3(r) = \int_a^r 2c^2s^{-5}\mathrm{H}^{-1}\,\mathrm{d}s, \tag{4.4.27}$$

and we clearly see that $\nabla G_2(r) = 2c^2r^{-6}\mathrm{H}^{-1}x$. Thus u_k, defined by g_k which is given by (4.4.14), is a solution to the full Euler-Lagrange given by (4.4.8). $\square$

4.5 Minimality of twist mappings in the homotopy classes $\mathscr{A}_k$ and their uniqueness

In this final section we aim to prove the main result of this paper. Namely that u_k, found in Section 4.4, is the unique minimiser of $\mathscr{F}$ in $\mathscr{A}_k$ for each $k \in \mathbb{Z}$, where $\mathscr{F}$ is defined by $\mathrm{F} \in \mathfrak{F}$ (a full description of $\mathfrak{F}$ is given below).

Our approach is to show that the symmetrisation $\bar{u} \in \mathscr{A}_k$, of $u \in \mathscr{A}_k$, strictly decreases the $\mathscr{F}$ energy if u is a non-twist mapping. To achieve this we rely on a lifting result for our mappings $u \in \mathcal{A}(\mathbf{X})$ which is given by Theorem 4.5.2. This result allows us to decompose our energies $\mathscr{F}$ in disjoint energy terms involving $|u|$ and some $g \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$. Focusing on the terms containing $|u|$ we prove, using the coarea formula, Propositions 4.5.1 and Proposition 4.5.2, that $|x|$ has less energy. Here we essentially reduce the problem to proving energy inequalities between $|u|$ and $|x|$ over almost every circle. It is then a consequence of the incompressibility constraint that equality can only occur if u is a twist mapping. The final stage of the proof is to show, via a suitable averaging argument, that there exists a $\bar{g} \in \mathscr{G}_k$ such that the energy terms involving g are decreased. Therefore, we conclude that the $\mathscr{F}$-energy of the twist mapping $\bar{u} = \mathrm{Q}[\bar{g}]x \in \mathscr{A}_k$ is strictly less energy than the $\mathscr{F}$-energy of $u \in \mathscr{A}_k$.

A consequence of this is that any minimiser of $\mathscr{F}$ in $\mathscr{A}_k$ must be a twist mapping. Hence we will obtain that u_k is the unique minimiser of $\mathscr{F}$ in $\mathscr{A}_k$ as u_k is the unique minimiser amongst twist mappings.

Before we proceed to the main result of this section let us firstly prove the preliminary results, alluded to above, which are needed for the proof of the main symmetrisation result. Namely our first result shows how the integral over the level set $\{x \in \mathbf{X} : |u| = t\}$ for some $u \in \mathcal{A}(\mathbf{X})$ and *a.e.* $t \in [a, b]$ is related to an integral of the inverse mapping $w = u^{-1} \in \mathcal{A}(\mathbf{X})$.

Proposition 4.5.1. *Let $u \in \mathcal{A}(\mathbf{X})$ and $w = u^{-1} \in \mathcal{A}(\mathbf{X})$ denote the inverse mapping of u. Then for $0 \leq \varphi \in \mathbf{C}[a^2, b^2]$ and for a.e. $t \in [a, b]$ we have the following identity,*

$$\int_0^{2\pi} \varphi(|w|^2) \left| \frac{\partial w}{\partial \theta} \right| \mathrm{d}\theta \bigg|_{r=t} = \int_{\{|u|=t\}} \varphi(|x|^2) \, \mathrm{d}\mathcal{H}^1, \tag{4.5.1}$$

where $\{|u| = t\} = \{x \in \mathbf{X} : |u| = t\}$ and $\partial w/\partial\theta = \nabla w \, x^\perp$ with $x^\perp = (-x_2, x_1)^t$.

Proof. To prove this result we rely on a version of the coarea formula for Sobolev functions as in [17] Proposition 2.1. Firstly let us fix a $t \in (a, b)$ and let us also take $0 < \delta < b - t$. Now define the open set $A = w(\mathbf{X}_t^{t+\delta}) \subset \mathbf{X}$ where $w = u^{-1}$ is the inverse of some fixed

$u \in \mathcal{A}(\mathbf{X})$ (*see* Theorem 4.2.1) and $\mathbf{X}_t^{t+\delta} = \{x \in \mathbf{X} : t < |x| < t + \delta\}$. Furthermore we shall let χ_A denote the characteristic function of A and therefore $\varphi_A(x) = \varphi(|x|^2)\chi_A(x)$ is a non-negative Borel measureable function. Hence by the coarea formula for Sobolev function (*see* [17]) ,

$$\int_{\mathbf{X}} \varphi_A |\nabla |u|| \,\mathrm{d}x = \int_a^b \int_{\{|u|=s\}} \varphi_A \,\mathrm{d}\mathcal{H}^1 \,\mathrm{d}s. \tag{4.5.2}$$

Moreover we note that the left hand side of (4.5.2) is given by,

$$\begin{aligned}\int_{\mathbf{X}} \varphi_A |\nabla |u|| \,\mathrm{d}x &= \int_{w(\mathbf{X}_t^{t+\delta})} \varphi(|x|^2)|\nabla |u|| \,\mathrm{d}x \\ &= \int_{\mathbf{X}} |y|^{-1}\varphi(|w|^2)|(\nabla w)^{-t}y| \,\mathrm{d}y. \end{aligned} \tag{4.5.3}$$

Above the last equality comes from an application of the change of variable formula given by (4.2.5) and the following calculation,

$$|u|(w(y)) = |y| \implies \nabla w(y)^t \nabla |u|(w) = |y|^{-1}y, \tag{4.5.4}$$

for *a.e.* $y \in \mathbf{X}$. Furthermore specifically in two dimensions we have that,

$$|(\nabla w)^{-t}y| = \left|\frac{\partial w}{\partial \theta}\right|, \tag{4.5.5}$$

and when combined with (4.5.2) we obtain that,

$$\int_t^{t+\delta} \int_0^{2\pi} \varphi(|w|^2)\left|\frac{\partial w}{\partial \theta}\right| \mathrm{d}\theta \,\mathrm{d}r = \int_a^b \int_{\{|u|=s\}} \varphi_A \,\mathrm{d}\mathcal{H}^1 \,\mathrm{d}s. \tag{4.5.6}$$

Moreover as u is a homeomorphism we also have that $\{|u| = s\} = w(\{|x| = s\})$ since,

$$\{x \in \mathbf{X} : |u| > s\} = w(\mathbf{X}_s), \tag{4.5.7}$$

where $\mathbf{X}_s = \{x \in \mathbf{X} : |x| > s\}$. Then from the uniform continuity of w on $\overline{\mathbf{X}}_s$ we see that,

$$w(\overline{\mathbf{X}}_s) = \overline{w(\mathbf{X}_s)}, \quad w(\partial \mathbf{X}_s) = \partial w(\mathbf{X}_s). \tag{4.5.8}$$

This therefore gives, using the identity boundary conditions, that

$$\partial\{x \in \mathbf{X} : |u| > s\} = \{x \in \mathbf{X} : |x| = b\} \cup w(\{x \in \mathbf{X} : |x| = s\}), \tag{4.5.9}$$

and in particular,

$$w(\{x \in \mathbf{X} : |x| = s\}) = \{x \in \mathbf{X} : |u| = s\}. \tag{4.5.10}$$

Therefore using this identity we have that for any $y \in \{|u| = t\}$ the characteristic function χ_A is non-zero at y, i.e. $\chi_A(y) \neq 0$, *iff* $t < s < t + \delta$. Hence (4.5.6) becomes,

$$\int_t^{t+\delta} \int_0^{2\pi} \varphi(|w|^2)\left|\frac{\partial w}{\partial \theta}\right| \mathrm{d}\theta \,\mathrm{d}r = \int_t^{t+\delta} \int_{\{|u|=s\}} \varphi \,\mathrm{d}\mathcal{H}^1 \,\mathrm{d}s. \tag{4.5.11}$$

Then since both integrands are L^1 integrable on (a, b) we obtain the result by an application of Lebesgue differentiation theorem. $\square$

In addition to the above stated proposition we shall also require in the proof of our main result the following proposition which was proved by the authors in [58]. Namely,

Proposition 4.5.2. *Let $\Gamma \in \mathbf{C}^2[a,b]$ be such that $\dot{\Gamma}(t)/t$ is a monotone increasing function. Then for $u \in \mathcal{A}(\mathbf{X})$ and almost every $r \in [a,b]$ we have that*

$$\int_0^{2\pi} \Gamma(|u|)\frac{(u(r,\theta)\times u_\theta(r,\theta))}{|u|^2}\,\mathrm{d}\theta \geq 2\pi\Gamma(r). \tag{4.5.12}$$

Proof. The proof of this result is given during the proof of Proposition 7.4 [58]. □

Our final result in this section before the main theorem is the following lifting result for mappings $u \in \mathcal{A}(\mathbf{X})$.

Theorem 4.5.3. *For each $u \in \mathscr{A}_k$ there exists a corresponding function $g \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$ such that u has the following lifting,*

$$u(x) = |u|(x)\mathrm{Q}[g]\frac{x}{|x|}, \tag{4.5.13}$$

and $g = 0, g = 2\pi k$ on $\partial\mathbf{X}_a$ and $\partial\mathbf{X}_b$ respectively. Furthermore,

$$|\nabla u|^2 = |\nabla|u||^2 + |x|^{-2}|u|^2\left(1 + 2\partial_\theta g\right) + |u|^2|\nabla g|^2, \tag{4.5.14}$$

for almost every $x \in \mathbf{X}$. Above we have used the following notation $\partial_\theta g = \nabla g \cdot x^\perp$ with $x^\perp = (-x_2, x_1)$.

Proof. Firstly as $u \in \mathcal{A}(\mathbf{X})$ we know that we can work with a representative $u \in \mathscr{C}(\mathbf{X})$ and furthermore $|u|^{-1}u \in W^{1,2}(\mathbf{X}, \mathbb{S}^1) \cap \mathbf{C}(\overline{\mathbf{X}}, \mathbb{S}^1)$. Now define,

$$v(r,\theta) = \frac{u}{|u|}(r\cos\theta, r\sin\theta) : [a,b]\times[0,2\pi] \to \mathbb{S}^1, \tag{4.5.15}$$

which due to $u/|u| \in W^{1,2}(\mathbf{X}, \mathbb{S}^1) \cap \mathbf{C}(\mathbf{X}, \mathbb{S}^1)$ gives that $v \in W^{1,2}(\mathcal{R}, \mathbb{S}^1) \cap \mathbf{C}(\mathcal{R}, \mathbb{S}^1)$ where $\mathcal{R} = [a,b]\times[0,2\pi]$ and $v(r,0) = v(r,2\pi)$. As $\mathcal{R}$ is simply connected we know from Theorem 3 in [15] that v has a lifting of the same regularity, i.e. $\exists h \in W^{1,2}(\mathcal{R}) \cap \mathbf{C}(\mathcal{R})$ such that $v(r,\theta) = e^{i(h(r,\theta)+\theta)}$. Now let $g(x) = g(r\cos\theta, r\sin\theta) = h(r,\theta)$ then we see that,

$$v(r,\theta) = e^{i(h(r,\theta)+\theta)} = \mathrm{Q}[g(r\cos\theta, r\sin\theta)]\frac{x}{|x|}, \tag{4.5.16}$$

and therefore $|u|^{-1}u(x) = |x|^{-1}\mathrm{Q}[g]x$. Since $g(r\cos\theta, r\sin\theta) = h(r,\theta)$ we know that g can only be non-continuous if there is a jump in h when $\theta = 0$ and $\theta = 2\pi$, namely $h(r,0) \neq h(r,2\pi)$. However we know that such a jump must be of $2\pi k$ for some $k \in \mathbb{Z}$ as $v(r,0) = v(r,2\pi)$. Therefore,

$$h(r,2\pi) = h(r,0) + 2\pi k, \tag{4.5.17}$$

which would imply for each $r \in [a,b]$ that the curve

$$\gamma_r(\theta) = v(r,\theta) = \frac{u}{|u|}(r\cos\theta, r\sin\theta), \tag{4.5.18}$$

has winding number $k+1$. However we know that the winding number of γ_r is 1 for each $r \in [a,b]$ due to the identity boundary conditions, i.e. $\gamma_a(\theta) = \gamma_b(\theta) = (\cos\theta, \sin\theta)$, which implies that $k = 0$. Hence $u = |u|\mathrm{Q}[g]|x|^{-1}x$ where $|u|, g \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$. Note that the identity boundary condition also gives that,

$$g = 2\pi k_a, \qquad g = 2\pi k_b, \tag{4.5.19}$$

on $\partial\mathbf{X}_a$ and $\partial\mathbf{X}_b$ respectively, where $k_a, k_b \in \mathbb{Z}$. Now without loss of generality we can assume that $k_a = 0$, as we can always take away an integer multiple of 2π from g such that this is the case. Then $u \in \mathscr{A}_k$ implies that $\mathbf{deg}(u) = k$ where here again the $\mathbf{deg}(u)$ is the winding number of the image under $|u|^{-1}u$ of any radial line (r,θ) for $\theta \in [0, 2\pi]$ fixed. This in turn implies that $g \equiv k_b = k$ on $\partial\mathbf{X}_b$. Finally (4.5.14) comes from a direct calculation which completes the proof. □

The significance of this result for the purpose of the proving the main result is the decomposition of $|\nabla u|^2$. This allows us to write the $\mathscr{F}$ energies into a series of functions energies, which themselves are easier to tame. In particular, we are able to apply an averaging argument to the angle of rotation function g in order to obtain a corresponding radial function $\bar{g}$, which in turn defines a twist mapping $\bar{u}$.

With these propositions now at hand we are in a position to prove the main result of this section. However let us first state the assumption we place on F.

Assumptions placed on the F defining the energies F

Firstly we assume, as stated before that $0 < c \leq \mathrm{F} \in \mathbf{C}^\infty(\mathscr{R})$. Moreover we assume that F also satisfies either of the following,

H1 $\mathrm{F}(x,y) = \mathrm{F}(1,y)x^{-1}$ and $\partial_y^2[\mathrm{F}(x^2,y^2)^{1/2}] \geq -\partial_y[\mathrm{F}(x^2,y^2)^{1/2}/y]$ for $y \in [a,b]$.

H2 $\mathrm{F}(x,y) = \mathrm{F}(x,1)y^{-1}$ and $\partial_x^2[\mathrm{F}(x^2,y^2)^{1/2}] \geq -\partial_x[\mathrm{F}(x^2,y^2)^{1/2}/x]$ for $x \in [a,b]$.

Then the class of functions we are interested in is,

$$\mathfrak{F} = \Big\{\mathrm{F} \in \mathbf{C}^\infty(\mathscr{R}) : \mathrm{F}(x,y) \text{ satisfies } (\mathbf{H1}) \text{ or } (\mathbf{H2})\Big\}. \tag{4.5.20}$$

For a specific example of a function $\mathrm{F}(x, y)$ which lies in $\mathfrak{F}$ consider $\mathrm{F}(x, y) = \varphi^2(x)/y$ where $\varphi(x) = x^\alpha$ on $[a, b]$ and $\alpha^2 \geq 1/4$. Then $\mathrm{F}(x, y)$ will lie in $\mathfrak{F}$ as it satisfies **H2**. With these examples in mind lets now consider the energies given by

$$\mathscr{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2) |\nabla u|^2 \, \mathrm{d}x \tag{4.5.21}$$

where $\mathrm{F} \in \mathfrak{F}$. Then we have the following result,

Theorem 4.5.4. *For any non-twist mapping $u \in \mathscr{A}_k$ there exists a twist mapping $\bar{u} \in \mathscr{A}_k$ which has strictly less $\mathscr{F}$ energy. Namely,*

$$\mathscr{F}[\bar{u}; \mathbf{X}] < \mathscr{F}[u; \mathbf{X}], \tag{4.5.22}$$

for each $k \in \mathbb{Z}$. Furthermore u_k is the unique minimiser of $\mathscr{F}$ over $\mathscr{A}_k$ for each $k \in \mathbb{Z}$.

Proof. Firstly we note that in proving this result it is enough to show that it holds for F just satisfying one of **H1** or **H2**, since the result for F satisfying the other condition follow from Theorem 4.2.1. Namely Theorem 4.2.1 shows that if $u \in \mathscr{A}_k$ then it's inverse exists and $w = u^{-1} \in \mathscr{A}_{-k}$ where in particular we have that,

$$|\nabla u(w(y))|^2 = |\nabla w(y)|^2, \tag{4.5.23}$$

for almost every $y \in \mathbf{X}$. Therefore an application of Theorem 1.8 on page 280 in [35] gives that,

$$\begin{aligned} \mathscr{F}[u; \mathbf{X}] &= \frac{1}{2} \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2) |\nabla u(x)|^2 \, \mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbf{X}} \mathrm{F}(|w(y)|^2, |y|^2) |\nabla w(y)|^2 \, \mathrm{d}y. \end{aligned} \tag{4.5.24}$$

Now let us define $\mathrm{F}^*(\xi, \eta) = \mathrm{F}(\eta, \xi)$ then clearly if F satisfies **H1** then F^* will satisfy **H2**. Moreover let us denote $\mathscr{F}^*$ to be,

$$\mathscr{F}[w; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \mathrm{F}^*(|x|^2, |w|^2) |\nabla w(x)|^2 \, \mathrm{d}x. \tag{4.5.25}$$

Therefore (4.5.24) gives that,

$$\mathscr{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \mathrm{F}^*(|y|^2, |w|^2) |\nabla w|^2 \, \mathrm{d}y = \mathscr{F}^*[w; \mathbf{X}]. \tag{4.5.26}$$

Hence showing that (4.5.22) holds for $\mathscr{F}$ gives that $\mathscr{F}^*[w; \mathbf{X}] > \mathscr{F}^*[\bar{u}^{-1}; \mathbf{X}]$. Additionally as $\bar{u}^{-1}$ is a twist mapping we obtain the result for $\mathscr{F}^*$.

In light of the above we shall assume that $\mathrm{F} \in \mathfrak{F}$ satisfies **H2**. Recall that from Theorem 4.5.3 we know that $\exists g \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$ corresponding to $u \in \mathscr{A}_k$. In particular,

$$|\nabla u|^2 = |\nabla |u||^2 + |x|^{-2}|u|^2\,(1 + 2\partial_\theta g) + |u|^2|\nabla g|^2, \tag{4.5.27}$$

for almost every $x \in \mathbf{X}$. Then using this identity we obtain that,

$$\begin{aligned} 2\mathscr{F}[u;\mathbf{X}] &= \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla u(x)|^2\,\mathrm{d}x \\ &= \int_{\mathbf{X}} \mathrm{F}(|x|^2, 1)|u|^{-2}|\nabla u(x)|^2\,\mathrm{d}x \\ &= \int_{\mathbf{X}} \mathrm{F}(|x|^2, 1)\left[|u|^{-2}|\nabla |u||^2 + |x|^{-2}\,(1 + 2\partial_\theta g) + |\nabla g|^2\right]\,\mathrm{d}x. \end{aligned} \tag{4.5.28}$$

Clearly the following holds,

$$\int_{\mathbf{X}} \mathrm{F}(|x|^2, 1)|x|^{-2}\,(1 + 2\partial_\theta g)\,\mathrm{d}x = \int_{\mathbf{X}} \mathrm{F}(|x|^2, |x|^2)\,\mathrm{d}x. \tag{4.5.29}$$

Therefore we are left with the following two terms to consider,

$$\mathbf{I} = \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla |u||^2\,\mathrm{d}x, \quad \mathbf{II} = \int_{\mathbf{X}} \mathrm{F}(|x|^2, 1)|\nabla g|^2\,\mathrm{d}x, \tag{4.5.30}$$

which for the rest of the proof we shall deal with individually in three steps after which we shall conclude the uniqueness of twist minimiser $u_k \in \mathscr{A}_k$ in one last step.

Step 1: Bounding I Below: Here we are considering **I** with the aim of trying to show the following,

$$\int_{\mathbf{X}} \mathrm{F}(|x|^2, |x|^2)\,\mathrm{d}x \le \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla |u||^2\,\mathrm{d}x. \tag{4.5.31}$$

The idea here is to use a form of the coarea formula from which we can use the propostions proved earlier in this section to gain inequalities on the boundary of the level sets, which in turn will allow us to obtain (4.5.31). To this end we first observe that for $u \in \mathcal{A}(\mathbf{X})$ the coarea formula for Sobolev functions (*see* [17]) gives that,

$$\int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)^{1/2}|\nabla |u||\,\mathrm{d}x = \int_a^b \int_{\{|u|=t\}} \mathrm{F}(|x|^2, t^2)\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t, \tag{4.5.32}$$

which then as $u \in \mathcal{A}(\mathbf{X})$ is a homeomorphism, due to Theorem 4.2.1, we saw from Proposition 4.5.1 that,

$$\int_0^{2\pi} \mathrm{F}(|w|^2, t^2)^{1/2}\left|\frac{\partial w}{\partial \theta}\right|\mathrm{d}\theta\bigg|_{r=t} = \int_{\{|u|=t\}} \mathrm{F}(|x|^2, t^2)^{1/2}\,\mathrm{d}\mathcal{H}^1, \tag{4.5.33}$$

for *a.e.* $t \in (a,b)$ where w denotes the inverse mapping of u. Furthermore we recall the straightforward identity,

$$\left|\frac{\partial w}{\partial \theta}\right|^2 = \frac{(w \cdot \partial_\theta w)^2}{|w|^2} + \frac{(w \times \partial_\theta w)^2}{|w|^2}. \tag{4.5.34}$$

This gives that $|\partial_\theta w| \geq (w \times \partial_\theta w)/|w|$ and therefore,

$$\int_0^{2\pi} \mathrm{F}(|w|^2, t^2)^{1/2} \frac{w \times \partial_\theta w}{|w|} \,\mathrm{d}\theta \bigg|_{r=t} \leq \int_{\{|u|=t\}} \mathrm{F}(|x|^2, t^2)^{1/2} \,\mathrm{d}\mathcal{H}^1, \tag{4.5.35}$$

for *a.e.* $t \in (a, b)$. Now for $t \in (a, b)$ fixed let $\Gamma(s) = \mathrm{F}(s^2, t^2)^{1/2} s$ and so

$$\dot{\Gamma}(s)/s = \frac{\mathrm{F}(s^2, t^2)^{1/2}}{s} + \frac{d}{ds}[\mathrm{F}(s^2, t^2)^{1/2}], \tag{4.5.36}$$

we obtain that,

$$\begin{aligned} \frac{d}{ds}\left(\frac{\dot{\Gamma}}{s}\right) &= \frac{d}{ds}\left(\frac{\mathrm{F}(s^2, t^2)^{1/2}}{s} + \frac{d}{ds}\left[\mathrm{F}(s^2, t^2)^{1/2}\right]\right) \\ &= \frac{d}{ds}\left(\frac{\mathrm{F}(s^2, t^2)^{1/2}}{s}\right) + \frac{d^2}{ds^2}\left[\mathrm{F}(s^2, t^2)^{1/2}\right] \geq 0, \end{aligned} \tag{4.5.37}$$

by our assumption **H2**. Thus $\dot{\Gamma}(s)/s$ is monotone increasing and as $\Gamma \in \mathbf{C}^2[a, b]$ we have by Proposition 4.5.2 that,

$$\int_0^{2\pi} \Gamma(|w|) \frac{w \times \partial_\theta w}{|w|^2} \,\mathrm{d}\theta \bigg|_{r=t} \geq 2\pi\Gamma(t) = 2\pi t \mathrm{F}(t^2, t^2)^{1/2}, \tag{4.5.38}$$

for *a.e.* $t \in [a, b]$. This therefore allows us to conclude from (4.5.35) that for *a.e.* $t \in (a, b)$ we have,

$$2\pi t \mathrm{F}(t^2, t^2)^{1/2} \leq \int_{\{|u|=t\}} \mathrm{F}(|x|^2, t^2)^{1/2} \,\mathrm{d}\mathcal{H}^1. \tag{4.5.39}$$

Now recall that $\alpha_{|u|}(t)$ is the distribution function of $|u|$, i.e. $\alpha_{|u|}(t) = |\{x \in \mathbf{X} : |u(x)| > t\}| = \pi(b^2 - t^2)$. Additionally to this let us recall that $1 = \det \nabla u = \partial_1 u \times \partial_2 u$ for *a.e.* $x \in \mathbf{X}$. Moreover each weak derivative of u can be written as,

$$\partial_1 u = \partial_1 |u| \frac{u}{|u|} + |u| \partial_1 \left(\frac{u}{|u|}\right), \quad \partial_2 u = \partial_2 |u| \frac{u}{|u|} + |u| \partial_2 \left(\frac{u}{|u|}\right). \tag{4.5.40}$$

Then an explicit calculation shows that,

$$\begin{aligned} \partial_1 u \times \partial_2 u &= \left[\partial_1 |u| \frac{u}{|u|} + |u| \partial_1 \left(\frac{u}{|u|}\right)\right] \times \left[\partial_2 |u| \frac{u}{|u|} + |u| \partial_2 \left(\frac{u}{|u|}\right)\right] \\ &= \partial_1 |u| \left[u \times \partial_2 \left(\frac{u}{|u|}\right)\right] + \partial_2 |u| \left[u \times \partial_1 \left(\frac{u}{|u|}\right)\right] \\ &= \left[u \times \partial_2 \left(\frac{u}{|u|}\right), u \times \partial_1 \left(\frac{u}{|u|}\right)\right] \nabla |u|, \end{aligned} \tag{4.5.41}$$

for *a.e.* $x \in \mathbf{X}$. Thus the constraint that $\det \nabla u = 1$ *a.e.* in $\mathbf{X}$ means that $|\{x \in \mathbf{X} : |\nabla |u|| = 0\}| = 0$. Therefore again via an application of the coarea formula for Sobolev functions as in Lemma 2.3 of [17] we obtain in this case that,

$$\int_{\{x \in \mathbf{X} : |u| = t\}} \frac{1}{|\nabla |u||} \mathrm{d}\mathcal{H}^1 = -\frac{d}{dt} \alpha_{|u|}(t) = 2\pi t, \tag{4.5.42}$$

for $a.e.$ $t \in (a,b)$. Then by Holder's inequality and an application of (4.5.39) we get that,

$$2\pi \mathrm{F}(t^2,t^2)t \leq \int_{\{x\in\mathbf{X}:|u|=t\}} \mathrm{F}(|x|^2,t^2)|\nabla|u||\,\mathrm{d}\mathcal{H}^1, \tag{4.5.43}$$

for almost every $t \in (a,b)$. Hence combining this with (4.5.32) we obtain,

$$\int_{\mathbf{X}} \mathrm{F}(|x|^2,|x|^2)\,\mathrm{d}x = \int_a^b 2\pi\mathrm{F}(t^2,t^2)\,t\,\mathrm{d}t \leq \int_{\mathbf{X}} \mathrm{F}(|x|^2,|u|^2)|\nabla|u||^2\,\mathrm{d}x, \tag{4.5.44}$$

and therefore we have proved (4.5.31).

Step 2: Case of equality with I: We now note that equality in (4.5.31) can only occur if u is a twist mapping. The basic idea here is that the steps taken above necessitate that $|u| = |x|$ $a.e.$ on $\mathbf{X}$ if we have equality, which by the determinant constraint and identity boundary conditions results in u having to be a twist. With this in mind we note that by the above calculations equality can only occur if

$$\left|\frac{\partial w}{\partial \theta}\right| = \frac{w \times \partial_\theta w}{|w|}, \tag{4.5.45}$$

for $a.e.$ $x \in \mathbf{X}$, which in turn implies that for almost every $x \in \mathbf{X}$ we have that,

$$\frac{\partial}{\partial\theta}|w|^2 = 2w \cdot \frac{\partial w}{\partial \theta} = 0. \tag{4.5.46}$$

Now recall from (4.5.4) that,

$$|y|\frac{\partial w}{\partial\theta}(y) = \mathrm{J}\nabla|u|(w), \quad \mathrm{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{4.5.47}$$

which gives by (4.5.46) that,

$$2w \cdot \mathrm{J}\nabla|u|(w) = 0, \implies \frac{\partial}{\partial\theta}|u|(x) = 0, \tag{4.5.48}$$

for $a.e.$ $x \in \mathbf{X}$. The last equality above comes from using that w is inverse of u and thus we let $y = u(x)$. We can deduce from this that $|u|(x) = h(|x|)$ for some $h \in L^2[a,b]$ and furthermore $\partial_r|u| = l(|x|)$ for some $l \in L^2[a,b]$. To see this is the case we first transfer into polar co-ordinates by letting $f(r,\theta) = |u|(r\cos\theta, r\sin\theta)$ then $f \in W^{1,2}(\mathcal{R})$ where $\mathcal{R} = [a,b] \times [0,2\pi]$. Moreover,

$$\frac{\partial f}{\partial r} = \frac{\partial|u|}{\partial r}, \qquad \frac{\partial f}{\partial\theta} = \frac{\partial|u|}{\partial\theta} = 0, \tag{4.5.49}$$

for almost every $(r,\theta) \in \mathcal{R}$. Therefore to prove our desired claim it is enough to show that $f(r,\theta) = h(r)$ for some $h \in L^2(a,b)$ and furthermore $\partial_r|f| = l(r)$ for some $l \in L^2(a,b)$. To this end we note that $\partial_\theta f = 0$ implies that,

$$0 = \int_{\mathcal{R}} f(r,\theta)\partial_\theta\varphi\,\mathrm{d}r\,\mathrm{d}\theta, \tag{4.5.50}$$

for all $\varphi \in \mathbf{C}_c^\infty(\mathcal{R})$. Now take any $\varphi \in \mathbf{C}_c^\infty(\mathcal{R})$ and define,

$$\psi(r,\theta) = \int_0^\theta \varphi(r,\xi)\,\mathrm{d}\xi - \theta \int_0^{2\pi} \varphi(r,\xi)\,\mathrm{d}\xi. \tag{4.5.51}$$

Therefore $\psi \in \mathbf{C}_c^\infty(\mathcal{R})$ and we obtain that,

$$\begin{aligned} 0 &= \int_{\mathcal{R}} f(r,\theta)\partial_\theta \psi \,\mathrm{d}r\,\mathrm{d}\theta = \int_{\mathcal{R}} f\left[\varphi(r,\theta) - \int_0^{2\pi}\varphi(r,\xi)\,\mathrm{d}\xi\right]\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_{\mathcal{R}} \left[f(r,\theta) - h(r)\right]\varphi(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta, \end{aligned} \tag{4.5.52}$$

for any $\varphi \in \mathbf{C}_c^\infty(\mathcal{R})$. In the last line we have let,

$$h(r) = \int_0^{2\pi} f(r,\theta)\,\mathrm{d}\theta, \tag{4.5.53}$$

and therefore $h \in L^2(a,b)$. Therefore as the above is true for any $\varphi \in \mathbf{C}_c^\infty(\mathcal{R})$ we can conclude, by the fundamental lemma of the calculus of variation, that $f(r,\theta) = h(r)$ for almost every $(r,\theta) \in \mathcal{R}$. In a similar fashion we can show that $\partial_r f = l(r)$ for some $l \in L^2(a,b)$ and almost every $(r,\theta) \in \mathcal{R}$. Now let us note the following,

$$\partial_r u \times \partial_\theta u = |x| \det \nabla u = |x|, \tag{4.5.54}$$

for almost every $x \in \mathbf{X}$. Additionally,

$$\partial_r u = \partial_r |u| \frac{u}{|u|} + |u| \partial_r \left(\frac{u}{|u|}\right), \tag{4.5.55}$$

$$\partial_\theta u = \partial_\theta |u| \frac{u}{|u|} + |u| \partial_\theta \left(\frac{u}{|u|}\right) = |u|\partial_\theta \left(\frac{u}{|u|}\right), \tag{4.5.56}$$

and therefore using the lifting result of Theorem 4.5.3 for $u \in \mathcal{A}(\mathbf{X})$ we can write the determinant constraint as,

$$2r = \frac{\partial |u|^2}{\partial r}\left(1 + \frac{\partial g}{\partial \theta}\right) \tag{4.5.57}$$

for *a.e.* $x \in \mathbf{X}$. Integrating both sides over $\mathbf{X}_r^{r+\delta}$ and using the $\partial_r |u|^2$ is constant with respect to θ we obtain that

$$\begin{aligned} 4\pi[(r+\delta)^3 - r^3] &= \int_r^{r+\delta} r\partial_r |u|^2 \int_0^{2\pi}\left(1 + \frac{\partial g}{\partial \theta}\right)\mathrm{d}\theta\,\mathrm{d}r \\ &= 2\pi \int_r^{r+\delta} r\partial_r |u|^2\,\mathrm{d}r. \end{aligned} \tag{4.5.58}$$

Then applying Lebesgue differentiation theorem we obtain that, $\partial_r |u|^2 = 2r$ for almost every $r \in [a,b]$ and thus $\partial_r |u|^2 = 2|x|$ for *a.e.* $x \in \mathbf{X}$. Moreover this gives that $|u| = |x|$ for all $x \in \mathbf{X}$ as $|u|$ and $|x|$ agree almost everywhere and are both continuous. Thus by (4.5.57) we know that for *a.e.* $x \in \mathbf{X}$,

$$\frac{\partial g}{\partial \theta} = 0, \tag{4.5.59}$$

from which it is possible to conclude, like earlier, that $g(x) = g(|x|)$ where $g \in W^{1,2}[a,b] \cap \mathbf{C}[a,b]$. Hence for equality to occur we have shown that,

$$u(x) = \mathrm{Q}[g(|x|)]x, \tag{4.5.60}$$

which is twist mapping. Thus equality can only hold for twist mappings. Thus we have shown that for a non-twist mapping u then,

$$\int_{\mathbf{X}} \mathrm{F}(|x|^2, |x|^2)\,\mathrm{d}x < \int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla|u||^2\,\mathrm{d}x. \tag{4.5.61}$$

Step 3: Bounding II Below: We are now left to deal with the **II** term. To this end we would like to show that,

$$\int_{\mathbf{X}} \mathrm{F}(|x|^2, 1)|\nabla \bar{g}|^2\,\mathrm{d}x \leq \int_{\mathbf{X}} \mathrm{F}(|x|^2, 1)|\nabla g|^2\,\mathrm{d}x,$$

for some $\bar{g} \in W^{1,2}(a,b)$ with $\bar{g}(a) = 0$ and $\bar{g}(b) = 2\pi k$. To achieve this we use an averaging argument. Namely let us define for $r \in (a,b)$,

$$\bar{g}(r) = \frac{1}{2\pi} \int_a^r \int_0^{2\pi} \frac{\partial g}{\partial r}(s,\theta)\,\mathrm{d}\theta\,ds, \tag{4.5.62}$$

where above we have written g in terms of polar co-ordinates. It is straightforward to see that $\bar{g}(a) = 0$, $\bar{g}(b) = 2\pi k$ and $\bar{g} \in W^{1,2}(a,b)$ since g satisfies these boundary conditions and $g \in W^{1,2}(\mathbf{X})$. Moreover an application of Jensen's inequality gives that,

$$|\nabla \bar{g}|^2 = \left|\frac{d\bar{g}}{dr}\right|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\nabla g|^2\,\mathrm{d}\theta, \tag{4.5.63}$$

for almost every $r \in (a,b)$. Thus we have that,

$$\int_{\mathbf{X}} \mathrm{F}(|x|^2, 1)|\nabla \bar{g}|^2\,\mathrm{d}x \leq \int_{\mathbf{X}} \mathrm{F}(|x|^2, 1)|\nabla g|^2\,\mathrm{d}x. \tag{4.5.64}$$

Hence combining all the estimates we have proved together we obtained that,

$$\begin{aligned} \mathscr{F}[u;\mathbf{X}] &= \frac{1}{2} \int_{\mathbf{X}} \mathrm{F}(|x|^2, 1) \left[|u|^{-2}|\nabla|u||^2 + |x|^{-2}\left(1 + 2g_\theta\right) + |\nabla g|^2\right]\,\mathrm{d}x \\ &\geq \frac{1}{2} \int_{\mathbf{X}} \mathrm{F}(|x|^2, |x|^2)|\nabla \bar{u}|^2 \mathrm{d}x = \mathscr{F}[\bar{u};\mathbf{X}], \end{aligned} \tag{4.5.65}$$

where due to (4.5.61) the inequality is strict if u is a non-twist mapping. Moreover by recalling $\bar{g}$ satisfies $\bar{g}(a) = 0$, $\bar{g}(b) = 2\pi k$ and $\bar{g} \in W^{1,2}(a,b)$ implies that the twist mapping $\bar{u} = \mathrm{Q}[\bar{g}]x \in \mathscr{A}_k$.

Step 4: Uniqueness of twist minimiser: Finally to conclude the proof we need to show that u_k is the unique minimiser of $\mathscr{F}$ in $\mathscr{A}_k$. To do this we suppose by contradiction this isn't the case, i.e. $\exists u \in \mathscr{A}_k$ such that $\mathscr{F}[u_k] = \mathscr{F}[u]$. Then as u_k is the unique minimiser amongst twist mapping in $\mathscr{A}_k$ we know that u must be a non-twist. However by the above we know that $\bar{u} \in \mathscr{A}_k$ satisfies,

$$\mathscr{F}[\bar{u}] < \mathscr{F}[u] = \mathscr{F}[u_k], \tag{4.5.66}$$

as u is a non-twist. Therefore we have reached our desired contradiction and conclude our proof. $\square$

Chapter 5

Incompressible twists as limits of compressible local energy minimisers on annuli: A Γ-convergence approach

Abstract

Consider a bounded annuli $\mathbf{X} \subset \mathbb{R}^n$ and the general stored energy functional given by,

$$\mathbb{E}_\varepsilon[u] = \int_{\mathbf{X}} \left[\frac{1}{2} \mathrm{F}(|x|^2, |u|^2) |\nabla u|^2 + \frac{1}{\varepsilon} h(\det \nabla u) \right] dx,$$

over the space of admissible mappings,

$$\mathcal{A}^+(\mathbf{X}) = \{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u > 0 \ a.e. \text{ and } u \equiv x \text{ on } \partial \mathbf{X}\}.$$

where $0 < \mathrm{F} \in \mathbf{C}^\infty$. Motivated by the previous work in [66], [74], [75], [76] and [78] we study the existence of equilibria to $\mathbb{E}_\varepsilon[u]$, namely classical solutions to the Euler-Lagrange associated with $\mathbb{E}_\varepsilon[u]$ over $\mathcal{A}^+(\mathbf{X})$, which are in the form of a twist mappings, i.e. $u(x) = f\mathrm{Q}[r]\theta$ where $\mathrm{Q} \in \mathbf{SO}(n)$ and $\theta = x/|x|$. In particular when $n = 2$ we show that there exists countable many critical points to $\mathbb{E}_\varepsilon[u]$ for each $\varepsilon > 0$ and moreover there are countable many twists L^1-local minimiser of $\mathbb{E}_\varepsilon$ when F is restricted to a large subclass of function. Furthermore, when $n > 2$ we prove an equivalence condition for a twist u to be a critical point of $\mathbb{E}_\varepsilon[u]$. This in turn shows that there are at least countable many critical twists mappings when n is *even* and at least one when n is *odd*.

Contents

5.1 Introduction

The focus of this paper is on the following class of deformations

$$\mathcal{A}^+(\mathbf{X}) = \{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u > 0 \ a.e. \text{ and } u \equiv x \text{ on } \partial\mathbf{X}\}, \tag{5.1.1}$$

of an n dimensional annulus $\mathbf{X} \subset \mathbb{R}^n$. In particular, we look at the associated deformation energy given to each $u \in \mathcal{A}^+(\mathbf{X})$ by,

$$\mathbb{E}[u; \mathbf{X}] = \int_{\mathbf{X}} W(x, u, \nabla u)\, dx, \tag{5.1.2}$$

where above $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}_+$ is a stored-energy function. For the purpose of this paper we assume that the stored-energy function is given by the prototypical polyconvex example of,

$$W(x, y, \mathbf{F}) = \frac{1}{2}\mathrm{F}(|x|^2, |y|^2) tr\{\mathbf{F}^t\mathbf{F}\} + \frac{1}{\varepsilon} h(\det \mathbf{F}). \tag{5.1.3}$$

Throughout the course of this paper we shall assume that $0 < \mathrm{F} \in \mathbf{C}^\infty(\mathscr{R})$ where $\mathscr{R} = (0, \infty) \times (0, \infty)$ and $0 < \varepsilon \in \mathbb{R}$. Furthermore, we place a number of assumptions on the function h, which are enumerated bellow.

[A1] $h \in \mathbf{C}^2(0, \infty)$ and strictly convex.

[A2] It obtains its unique global minimum of zero at one, i.e. $h(1) = 0$.

[A3] There exists $\xi > 0$ and $\eta > 0$ such that for all $t \in (0, \infty)$ and $\alpha > 0$ satisfying $|\alpha - 1| < \eta$ then,

$$|\dot{h}(\alpha t)t| \leq \xi[h(t) + 1]. \tag{5.1.4}$$

[A4] The following asymptotic behaviour is satisfied:

$$\lim_{t \searrow 0} h(t) = \infty. \tag{5.1.5}$$

Thus for each $\varepsilon > 0$, h satisfying [**A1**] – [**A4**] and $0 < \mathrm{F} \in \mathbf{C}^\infty(\mathscr{R})$ we have an associated deformation energy given by,

$$\mathbb{E}_\varepsilon[u; \mathbf{X}] = \int_{\mathbf{X}} \left[\frac{1}{2} \mathrm{F}(|x|^2, |u|^2) |\nabla u|^2 + \frac{1}{\varepsilon} h(\det \nabla u) \right] dx. \tag{5.1.6}$$

The Euler-Lagrange equations $\mathbb{EL}[u]$ formally associated with the deformation energy $\mathbb{E}_\varepsilon[u; \mathbf{X}]$ over $\mathcal{A}^+(\mathbf{X})$ at $u \in \mathcal{A}^+(\mathbf{X})$ is,

$$\begin{cases} \operatorname{div}(\mathrm{F} \nabla u) - \partial_2 \mathrm{F} |\nabla u|^2 u + \varepsilon^{-1} \nabla \left[\dot{h}(\det \nabla u) \operatorname{cof} \nabla u \right] = 0, & \text{in } \mathbf{X}, \\ \det \nabla u > 0, & \text{in } \mathbf{X}, \\ u \equiv x & \text{on } \partial \mathbf{X}, \end{cases} \tag{5.1.7}$$

where above we have let $\mathrm{F} = \mathrm{F}(|x|^2, |u|^2)$ and $\partial_2 \mathrm{F}(x, y) = \partial \mathrm{F} / \partial y$. A mapping $u \in \mathbf{C}^2(\mathbf{X}, \mathbb{R}^n) \cap \mathbf{C}(\overline{\mathbf{X}}, \mathbb{R}^n)$ with finite deformation energy $\mathbb{E}_\varepsilon[u; \mathbf{X}]$ and which satisfies (5.1.7) is called a *classical solution* or equilibria to the energy $\mathbb{E}_\varepsilon$ over $\mathcal{A}^+(\mathbf{X})$. Furthermore, a consequence of a classical solution u being $\mathbf{C}^2$ is that the system of Euler-Lagrange equations (5.1.7), which u satisfies, can be equivalently expressed using the so-called Piola identity and pointwise invertibility of the gradient matrix as

$$\begin{cases} (\nabla u)^t \left[\partial_2 \mathrm{F} |\nabla u|^2 u - \operatorname{div}(\mathrm{F} \nabla u) \right] = \varepsilon^{-1} \det \nabla u \nabla (\dot{h}(\det \nabla u)), & \text{in } \mathbf{X}, \\ \det \nabla u > 0, & \text{in } \mathbf{X}, \\ u \equiv x & \text{on } \partial \mathbf{X}. \end{cases} \tag{5.1.8}$$

Motivated by previous work in [66], [74], [75], [76] and [78] we set ourselves the task of studying under what conditions twist mappings provide equilibria to the energies $\mathbb{E}_\varepsilon$ over $\mathcal{A}^+(\mathbf{X})$. In this paper we define a twist mapping to have the form

$$u(x) = f(r) \mathrm{Q}[r] \theta, \tag{5.1.9}$$

where $r = |x|$, $\theta = x/|x|$, $\mathrm{Q} \in W^{1,2}([a, b], \mathbf{SO}(n))$ with $\mathrm{Q}[a] = \mathrm{Q}[b] = \mathbf{I}_n$ and $f \in W^{1,2}[a, b]$ with $f(a) = a$ and $f(b) = b$. To ensure that a twist mapping u given by (5.1.9) belongs to our class of admissible mappings $\mathcal{A}^+(\mathbf{X})$ it is enough to assume that $\dot{f} > 0$ *a.e.* on (a, b).

Through the course of the paper we shall see, namely Theorem 5.3.1 and Theorem 5.4.1, that when n is *even* there exists a twist mapping $u_k \in \mathcal{A}^+(\mathbf{X})$ for every $k \in \mathbb{Z}$ which is a solution to (5.1.8). However when n is *odd* Theorem 5.3.1 and Theorem 5.4.1 tells us that only twists of the form $u = f(r)\theta$ are possible solutions to (5.1.8) and that there is atleast one solution. The existence of the twist solutions in both even and odd dimensional cases is obtained in Theorem 5.4.1 and requires the following additional assumption.

[**A5**] The following suplinear grow of h is satisfied.

$$\lim_{t\to\infty} h(t)/t = \infty. \tag{5.1.10}$$

A major theme running through this paper is a direct consequence of the aforementioned assumptions on h. Namely, assumptions [**A2**] and [**A4**] allow us to consider the incompressible model

$$\mathbb{F}[u;\mathbf{X}] = \int_{\mathbf{X}} \frac{1}{2}\mathrm{F}(|x|^2, |u|^2)|\nabla u|^2\, dx, \tag{5.1.11}$$

over the space of admissible mappings,

$$\mathcal{A}(\mathbf{X}) = \{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ and } u \equiv x \text{ on } \partial\mathbf{X}\}, \tag{5.1.12}$$

as the Γ-limit of $\mathbb{E}_\varepsilon$ as $\varepsilon \to 0$ (i.e. $\mathbb{E}_\varepsilon \xrightarrow{\Gamma} \mathbb{F}$). With this Γ-convergence in mind we study in the planar case $n = 2$ the convergence, with respect to $W^{1,2}$, of minimisers u^k_ε to $\mathbb{E}_\varepsilon$ in homotopy classes of $\mathcal{A}^+(\mathbf{X})$ as $\varepsilon \to 0$. In this setting it is possible to show that $\mathcal{A}^+(\mathbf{X})$ countably many pairwise disjoint homotopy classes $\mathscr{A}^+_k$ each containing a minimiser u^k_ε to $\mathbb{E}_\varepsilon$ for $\varepsilon > 0$. In Section 5.5 we prove that the minimisers in each homotopy class are converging strongly in $W^{1,2}$ to the unique minimisers u^k, in the respective homotopy class $\mathscr{A}_k$, for the limit energy $\mathbb{F}$. Uniqueness of minimisers in homotopy classes for the limit energy (the incompressible setting) was achieved by the authors in [60] where it was shown that the minimisers are in fact incompressible twist mappings. (Note that throughout this section additional assumptions are placed on F, i.e. (5.5.12) and (5.5.13))

The final two sections of this paper are devoted to the proof of our main result which is an analogue result to that found in [60] for the incompressible setting. Namely, we prove in the planar setting that minimisers in the homotopy classes of $\mathcal{A}^+(\mathbf{X})$ do exist in the form of a twist mappings. This result in turn gives that for any $\varepsilon > 0$ there are countably many L^1-local minimiser of the form of a twist mapping. Hence, for each $\varepsilon > 0$ and $k \in \mathbb{Z}$ there is a corresponding L^1-local minimisers of $\mathbb{E}_\varepsilon$ is given by,

$$u^k_\varepsilon(x) = f^k_\varepsilon \exp(2\pi k\beta\, \mathrm{J})\theta, \tag{5.1.13}$$

where (f_ε^k, β) are the minimisers of $\mathscr{E}[(f,\beta);[a,b]]$ which is defined by (5.4.5) and J is the 2×2 skew-symmetric matrix with ± 1 in the off-diagonal.

5.2 Twist mappings, the restricted energy $\mathbb{K}_\varepsilon$ and the reduced Euler-Lagrange system

Twist mappings have been well studied in the context of incompressible models (*see* [60], [58] or [75]). In this section we introduce twist mappings in the current context of compressible models, as in [66], and look at some of the fundamental properties and identities satisfied by twist mappings. Aside from introducing twist mappings the overall objective of this section is twofold: Firstly, we wish to introduce and define the restricted energy $\mathbb{K}_\varepsilon$ which occurs when the energy $\mathbb{E}_\varepsilon[u;\mathbf{X}]$ is restricted to twist mappings. Secondly, we wish to derive the reduce Euler-Lagrange system associated to the restricted energy $\mathbb{K}_\varepsilon$ over twist mappings. To this end let us first formal define twist mappings.

Definition 5.2.1. (Twist mapping) A twist mapping $u:\mathbf{X}\to\mathbb{R}^n$ of the n-dimensional annulus $\mathbf{X}\subset\mathbb{R}^n$ has the form (below $\theta = x/|x|$)

$$u(x) = f(r)\mathrm{Q}[r]\theta, \tag{5.2.1}$$

where $f\in W^{1,2}(a,b)$ such that $f(a)=a$ and $f(b)=b$. Whilst Q is a closed $W^{1,2}$ Sobolev loop starting from $\mathbf{I}_n$ in $\mathbf{SO}(n)$, i.e. $\mathrm{Q}\in W^{1,2}([a,b],\mathbf{SO}(n))$ with $\mathrm{Q}[a]=\mathrm{Q}[b]=\mathbf{I}_n$.

Moreover we let $\mathscr{T}\subset W^{1,2}(\mathbf{X},\mathbb{R}^n)$ denote the set of n-dimensional twists. It is immediate, from the above definition, that $u(x)=x$ on $\partial\mathbf{X}$ for any $u\in\mathscr{T}$. Furthermore, twist mappings satisfy some useful identities which for the convenience of the reader we collect and state as a proposition. Namely,

Proposition 5.2.1. *Any (twist) mapping $u\in\mathscr{T}$ satisfies the following identities,*

(i) $\nabla u = r^{-1}f\mathrm{Q}[r] + (\dot f - r^{-1}f)\mathrm{Q}\theta\otimes\theta + f\dot{\mathrm{Q}}\theta\otimes\theta$,

(ii) $\det\nabla u = r^{-(n-1)}\dot f f^{n-1}$,

(iii) $|\nabla u|^2 = (n-1)r^{-2}f^2 + \dot f^2 + f^2|\dot{\mathrm{Q}}\theta|^2$.

Additionally if f and Q satisfies further differentiability assumptions then,

(iv) $\Delta u = \left[\alpha\mathrm{Q} + \beta\dot{\mathrm{Q}} + f\ddot{\mathrm{Q}}\right]\theta$,

where we have let $\alpha = \ddot f + r^{-1}(n-1)(\dot f - r^{-1}f)$ and $\beta = 2\dot f + r^{-1}(n-1)f$.

Proof. These can be verified by direct calculations as in [66]. $\square$

A consequence of Proposition 5.2.1 is the following corollary which asserts that a general twist mapping belongs to our admissible class of maps *iff* the radial part satisfies $\dot{f} > 0$ *a.e.* on (a, b).

Corollary 5.2.1. A (twist) mapping $u \in \mathscr{T}$ satisfies $u \in \mathcal{A}^+(\mathbf{X})$ if and only if $\dot{f}(r) > 0$ for *a.e.* $r \in (a, b)$.

Proof. Firstly (iii) of Proposition 5.2.1 gives that $u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n)$ and by (ii) $\det \nabla u > 0$ *a.e.* if and only if $\dot{f}(r) > 0$ for *a.e.* $r \in (a, b)$. $\square$

With the above corollary in mind we fix some further notation. Namely let,

$$\mathscr{T}_+ := \left\{ u \in \mathscr{T} : \frac{d|u|}{dr}(r) = \dot{f}(r) > 0 \ a.e. \right\}, \tag{5.2.2}$$

which by Corollary 5.2.1 satisfies $\mathscr{T}_+ \subset \mathcal{A}^+(\mathbf{X})$. Now by considering the energy $\mathbb{E}_\varepsilon[u; \mathbf{X}]$ over twist mappings $u \in \mathcal{A}^+(\mathbf{X})$, i.e. restricting $\mathbb{E}_\varepsilon[u; \mathbf{X}]$ to $u \in \mathscr{T}_+$, we find that,

$$\begin{aligned} \mathbb{E}_\varepsilon[u; \mathbf{X}] &= \int_{\mathbf{X}} \frac{\mathrm{F}(r^2, f^2)}{2} \left[(n-1)\frac{f^2}{r^2} + \dot{f}^2 + f^2 |\dot{\mathrm{Q}}\theta|^2 \right] + \frac{1}{\varepsilon} h\left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) dx \\ &= \eta \int_a^b r^{n-1} \left\{ \mathrm{F} \left[(n-1)\frac{f^2}{r^2} + \dot{f}^2 + f^2 \frac{|\dot{\mathrm{Q}}|^2}{n} \right] + \frac{2}{\varepsilon} h\left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) \right\} dr \\ &=: \eta\, \mathbb{K}_\varepsilon[f, \mathrm{Q}], \end{aligned} \tag{5.2.3}$$

where above we have let $\mathrm{F} = \mathrm{F}(r^2, f^2)$ and $\eta = 2^{-1} n\omega_n$. Here ω_n denotes the volume of the n-dimensional unit ball, i.e. $\omega_n = \mathrm{Vol}(\mathbb{B}_n)$. Therefore, minimising $\mathbb{E}_\varepsilon[u; \mathbf{X}]$ over the class of twist mappings $\mathscr{T}_+$ is equivalent to minimising the restricted energy

$$\begin{aligned} \mathbb{K}_\varepsilon[f, \mathrm{Q}] = \int_a^b r^{n-1} \Bigg\{ \mathrm{F} \Bigg[(n-1)\frac{f^2}{r^2} \\ + \dot{f}^2 + f^2 \frac{|\dot{\mathrm{Q}}|^2}{n} \Bigg] + \frac{2}{\varepsilon} h\left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) \Bigg\} dr \end{aligned} \tag{5.2.4}$$

over $\mathscr{W} = \mathscr{F}_+ \times \mathscr{Q}$ where,

$$\mathscr{F}_+ = \{ f \in W^{1,2}(a, b) : \dot{f}(r) > 0 \ a.e. \text{ and } f(a) = a, \ f(b) = b \}, \tag{5.2.5}$$

and

$$\mathscr{Q} = \{ \mathrm{Q} \in W^{1,2}([a, b], \mathbf{SO}(n)) : \mathrm{Q}[a] = \mathrm{Q}[b] = \mathbf{I}_n \}. \tag{5.2.6}$$

Now that we have obtained the restricted energy associated to restricting $\mathbb{E}_\varepsilon[u; \mathbf{X}]$ over the space of twist mappings we can go about deriving the reduced Euler-Lagrange system $\mathbb{EL}[f, \mathrm{Q}]$.

Proposition 5.2.2. *Let* $(f,\mathrm{Q}) \in \mathscr{W}$ *have finite energy, i.e.* $\mathbb{K}_\varepsilon[f,\mathrm{Q}] < \infty$, *with* $\mathrm{Q} \in \mathbf{C}^2((a,b),\mathbf{SO}(n))$ *and* $f \in \mathbf{C}^2(a,b)$. *Then the Euler-Lagrange system* $\mathbb{EL}[f,\mathrm{Q}]$ *associated with* $\mathbb{K}_\varepsilon[f,\mathrm{Q}]$ *over the admissible pairs* $\mathscr{W}$ *at* (f,Q) *is*

$$\begin{cases} \dfrac{d}{dr}\left[r^{n-1}\mathrm{F}f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right] = 0, \\ \\ \dfrac{d}{dr}\left[r^{n-1}\mathrm{F}\dot{f}\right] + \dfrac{f^{n-1}}{\varepsilon}\dfrac{d}{dr}\dot{h} = fr^{n-1}\left[\partial_2\mathrm{F}L + \mathrm{F}\left(\dfrac{n-1}{r^2} + \dfrac{|\dot{\mathrm{Q}}|^2}{n}\right)\right]. \end{cases} \tag{5.2.7}$$

where above $\mathrm{F} = \mathrm{F}(r^2,f^2)$, $\dot{h} = \dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)$ *and*

$$L = L(f,|\dot{\mathrm{Q}}|) = (n-1)\frac{f^2}{r^2} + \dot{f}^2 + f^2\frac{|\dot{\mathrm{Q}}|^2}{n}. \tag{5.2.8}$$

Proof. The Euler-Lagrange system associated to the reduced energy (5.2.4) over $\mathscr{W}$ can be found by studying the variation with respect to $f \in \mathscr{F}_+$ and $\mathrm{Q} \in \mathscr{Q}$ separately. Firstly the variation with respect to $\mathrm{Q} \in \mathscr{Q}$ gives that,

$$\frac{d}{dr}\left[r^{n-1}\mathrm{F}(r^2,f^2)f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right] = 0. \tag{5.2.9}$$

This can be seen by taking a variation of the form $\mathrm{Q}_\delta = \mathrm{Q} + \delta\mathrm{Q}(\mathbb{B} - \mathbb{B}^t)$ where $\mathbb{B} \in \mathbf{C}_0^\infty((a,b),\mathbb{R}^{n\times n})$ as in [66]. Now consider the first variation with respect to $f \in \mathscr{F}_+$, i.e. $\frac{d}{d\delta}\mathbb{K}_\varepsilon[f+\delta\varphi,\mathrm{Q}]_{\delta=0} = 0$ where $\varphi \in \mathbf{C}_0^\infty(a,b)$. It is important to note that for each fixed $f \in \mathscr{F}_+$ we can find for each $\varphi \in \mathbf{C}_0^\infty(a,b)$ a $\delta > 0$ sufficiently small such that $\dot{f} + \delta\dot{\varphi} > 0$ for *a.e.* $r \in (a,b)$ and $\mathbb{K}_\varepsilon[f_\varepsilon,\mathrm{Q}] < \infty$. Therefore, (below we write $\mathrm{F} = \mathrm{F}(r^2,f^2)$)

$$\begin{aligned} 0 = {} & 2\int_a^b r^{n-1}\left[\partial_2\mathrm{F}fL(f,|\dot{\mathrm{Q}}|)\varphi + \mathrm{F}\left(\frac{n-1}{r^2}f\varphi + \dot{f}\dot{\varphi} + f\varphi\frac{|\dot{\mathrm{Q}}|^2}{n}\right)\right]dr \\ & + 2\int_a^b r^{n-1}\frac{1}{\varepsilon}\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)\left[\dot{\varphi}\frac{f^{n-1}}{r^{n-1}} + \frac{(n-1)\dot{f}}{r^{n-1}}f^{n-2}\varphi\right]dr. \end{aligned} \tag{5.2.10}$$

In the above we have let,

$$L(f,|\dot{\mathrm{Q}}|) = (n-1)\frac{f^2}{r^2} + \dot{f}^2 + f^2\frac{|\dot{\mathrm{Q}}|^2}{n}, \tag{5.2.11}$$

as in (5.2.8). Then rearranging (5.2.10) we find that,

$$\begin{aligned} 0 = {} & \int_a^b r^{n-1}f\left[\partial_2\mathrm{F}(r^2,f^2)L(f,|\dot{\mathrm{Q}}|) + \mathrm{F}(r^2,f^2)\left(\frac{n-1}{r^2} + \frac{|\dot{\mathrm{Q}}|^2}{n}\right)\right]\varphi dr \\ & + \int_a^b \frac{(n-1)\dot{f}f^{n-2}}{\varepsilon}\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)\varphi\, dr \\ & + \int_a^b r^{n-1}\left[\mathrm{F}(r^2,f^2)\dot{f} + \frac{1}{\varepsilon}\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)\left(\frac{f}{r}\right)^{n-1}\right]\dot{\varphi}dr. \end{aligned} \tag{5.2.12}$$

Hence integration by parts and an application of the fundamental lemma of the calculus of variation gives,

$$
\begin{aligned}
&\frac{d}{dr}\left\{ r^{n-1}\mathrm{F}(r^2,f^2)\dot{f} + \frac{f^{n-1}}{\varepsilon}\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)\right\} \\
&= f r^{n-1}\left[\partial_2\mathrm{F}(r^2,f^2)L(f,|\dot{\mathrm{Q}}|) + \mathrm{F}(r^2,f^2)\left(\frac{n-1}{r^2}+\frac{|\dot{\mathrm{Q}}|^2}{n}\right)\right] \\
&\qquad\qquad + \frac{(n-1)\dot{f}f^{n-2}}{\varepsilon}\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)
\end{aligned} \tag{5.2.13}
$$

Note through explicit differentiation we have that,

$$
\begin{aligned}
\frac{d}{dr}\left[\frac{f^{n-1}}{\varepsilon}\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)\right] &= \frac{(n-1)f^{n-2}\dot{f}}{\varepsilon}\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right) \\
&\qquad + \frac{f^{n-1}}{\varepsilon}\frac{d}{dr}\left[\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)\right].
\end{aligned} \tag{5.2.14}
$$

Hence by substituting the above into (5.2.13) we obtain,

$$
\begin{aligned}
&\frac{d}{dr}\left[r^{n-1}\mathrm{F}(r^2,f^2)\dot{f}\right] + \frac{f^{n-1}}{\varepsilon}\frac{d}{dr}\left[\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)\right] \\
&= f r^{n-1}\left[\partial_2\mathrm{F}(r^2,f^2)L(f,|\dot{\mathrm{Q}}|) + \mathrm{F}(r^2,f^2)\left(\frac{n-1}{r^2}+\frac{|\dot{\mathrm{Q}}|^2}{n}\right)\right],
\end{aligned} \tag{5.2.15}
$$

which therefore concludes our proof of the proposition. □

5.3 Equivalence condition for twist mappings to be equilibria of $\mathbb{E}_\varepsilon$

The purpose of this section is to prove an equivalence relationship for twist mapping to be classical solutions to the full Euler-Lagrange. The benefit of this equivalence relationship will become apparent in the next section when we prove the existence of twist mappings which are classical solutions to the full Euler-Lagrange system (5.1.8). An immediate outcome of the aforementioned equivalence result which is of interest however is that in odd dimensions a twist mapping $u \in \mathscr{T}_+ \cap \mathbf{C}^2(\mathbf{X}, \mathbb{R}^n)$ is a solution to (5.1.8), i.e. $\mathbb{EL}[u] = 0$, only if it has the form $u(x) = f(r)\theta$ where $f \in \mathscr{F}_+ \cap \mathbf{C}^2(a,b)$. This puts a serious restriction on the type of twist mappings which can be equilibria to $\mathbb{E}_\varepsilon[u;\mathbf{X}]$ over $\mathcal{A}^+(\mathbf{X})$.

Theorem 5.3.1. *A twist mapping $u \in \mathscr{T}_+$ where $\mathrm{Q} \in \mathbf{C}^2((a,b),\mathbf{SO}(n))$ and $f \in \mathbf{C}^2(a,b)$ with $(f,\mathrm{Q}) \in \mathscr{W}$ is a solution to the full Euler-Lagrange equation* (5.1.8) *when n is even* *if and only if,*

$$\begin{cases} \mathbb{EL}[f,\mathrm{Q}]=0, \\ \mathrm{Q}=P\exp(2\pi k\beta \mathrm{J})P^t, \end{cases} \tag{5.3.1}$$

for any $k\in\mathbb{Z}$ *and where* $\mathrm{J}=\mathrm{diag}(\mathrm{J}_2,\dots,\mathrm{J}_2)$,

$$\mathrm{J}_2=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \qquad \beta(r)=\frac{1}{c}\int_a^r\left[s^{n-1}f^2F(s^2,f^2)\right]^{-1}\,ds. \tag{5.3.2}$$

Moreover twist mapping $u\in\mathscr{T}_+$ *with* $f\in\mathscr{F}_+\cap\mathbf{C}^2(a,b)$ *is a solution to the full Euler-Lagrange in odd dimensions if and only if*

$$\begin{cases} \mathbb{EL}[f,\mathbf{I}_n]=0, \\ \mathrm{Q}=\mathbf{I}_n. \end{cases} \tag{5.3.3}$$

Above the constant c *denotes* $c=\int_a^b\left[r^{n-1}f^2F(r^2,f^2)\right]^{-1}\,dr$.

Remark 5.3.1. The above theorem essentially gives us a restriction on the solutions to reduced Euler-Lagrange $\mathbb{EL}[f,\mathrm{Q}]=0$ which can also be solutions to the full Euler-Lagrange $\mathbb{EL}[u]=0$. Namely we have the following equivalence for twist mappings $u\in\mathscr{T}_+$,

$$\mathbb{EL}[u]=0 \iff \mathbb{EL}[f,\mathrm{Q}]=0 \ \text{ and } \begin{cases} \mathrm{Q}=P\exp(2\pi k\beta\mathrm{J})P^t & n \text{ even}, \\ \mathrm{Q}=\mathbf{I}_n & n \text{ odd}. \end{cases}$$

for some $P\in\mathbf{SO}(n)$ and β given in (5.3.2). Let us also now remark that the proof of the above stated result is essentially a series of direct and substantial calculations.

Proof. To begin we note in the case that $u\in\mathscr{T}_+$ satisfies the full Euler-Lagrange $\mathbb{EL}[u]=0$ then $\mathbb{EL}[f,\mathrm{Q}]=0$. This is a consequence of the equation $\mathbb{EL}[u]=0$ resulting from considering a larger class of variations than in the $\mathbb{EL}[f,\mathrm{Q}]$ case. Hence in either situation of the theorem we know that (f,Q) satisfy the reduced Euler-Lagrange, i.e. $\mathbb{EL}[f,\mathrm{Q}]=0$. With this in mind we now make a note of some general identities satisfied by twist mappings. Firstly recalling (i) in Proposition 5.2.1 we find that,

$$\begin{aligned}(\nabla u)^t u &= f\left[\frac{f}{r}\mathrm{Q}^t+\left(\dot f-\frac{f}{r}\right)\theta\otimes\mathrm{Q}\theta+f\theta\otimes\dot{\mathrm{Q}}\theta\right]\mathrm{Q}\theta \\ &= \left[\frac{f^2}{r}+f\left(\dot f-\frac{f}{r}\right)\right]\theta=f\dot f\theta. \end{aligned}\tag{5.3.4}$$

Another application of (i) in Proposition 5.2.1 allows for the following expansion of

$(\nabla u)^t \nabla u$. Namely,

$$
\begin{aligned}
(\nabla u)^t \nabla u = & \left[\frac{f}{r}\mathrm{Q}^t + \left(\dot{f} - \frac{f}{r}\right)\theta \otimes \mathrm{Q}\theta + f\theta \otimes \dot{\mathrm{Q}}\theta\right] \times \\
& \times \left[\frac{f}{r}\mathrm{Q} + \left(\dot{f} - \frac{f}{r}\right)\mathrm{Q}\theta \otimes \theta + f\dot{\mathrm{Q}}\theta \otimes \theta\right] \\
= & \left[\frac{f^2}{r^2}\mathbf{I}_n + 2\frac{f}{r}\left(\dot{f} - \frac{f}{r}\right)\theta \otimes \theta + \frac{f^2}{r}\mathrm{Q}^t\dot{\mathrm{Q}}\theta \otimes \theta + \right. \\
& \left. + \left(\dot{f} - \frac{f}{r}\right)^2 \theta \otimes \theta + \frac{f^2}{r}\theta \otimes \mathrm{Q}^t\dot{\mathrm{Q}}\theta + f^2|\dot{\mathrm{Q}}\theta|^2\theta \otimes \theta\right].
\end{aligned}
\tag{5.3.5}
$$

The above identity (5.3.5) then gives that,

$$
\begin{aligned}
(\nabla u)^t \nabla u\, \theta &= \left[\frac{f^2}{r^2} + 2\frac{f}{r}\left(\dot{f} - \frac{f}{r}\right) + \left(\dot{f} - \frac{f}{r}\right)^2 + f^2|\dot{\mathrm{Q}}\theta|^2\right]\theta + \frac{f^2}{r}\mathrm{Q}^t\dot{\mathrm{Q}}\theta \\
&= \left[\dot{f}^2 + f^2|\dot{\mathrm{Q}}\theta|^2\right]\theta + \frac{f^2}{r}\mathrm{Q}^t\dot{\mathrm{Q}}\theta \\
&= L(f, \sqrt{n}|\dot{\mathrm{Q}}\theta|) + \frac{f^2}{r}\mathrm{Q}^t\dot{\mathrm{Q}}\theta - (n-1)\frac{f^2}{r^2}.
\end{aligned}
\tag{5.3.6}
$$

Note that as previously $L(f, \sqrt{n}|\dot{\mathrm{Q}}\theta|)$ is given by (5.2.8). Additionally recall that the full Euler-Lagrange system is given by,

$$
(\nabla u)^t \left[\partial_2 \mathrm{F}(|x|^2, |u|^2)|\nabla u|^2 u - \operatorname{div}(\mathrm{F}(|x|^2, |u|^2)\nabla u)\right] = \frac{\det \nabla u}{\varepsilon}\nabla\left(\dot{h}(\det \nabla u)\right),
$$

which for a twist mapping becomes the following, (note $|\nabla u|^2 = L(f, \sqrt{n}|\dot{\mathrm{Q}}\theta|)$ for twist mappings)

$$
\partial_2 \mathrm{F} L(f, \sqrt{n}|\dot{\mathrm{Q}}\theta|)(\nabla u)^t u - \frac{d\mathrm{F}}{dr}(\nabla u)^t \nabla u\, \theta - \mathrm{F}(\nabla u)^t \Delta u = \frac{\gamma}{\varepsilon}\frac{d}{dr}\left[\dot{h}(\gamma)\right]\theta. \tag{5.3.7}
$$

Above for simplicity we have used $\gamma(r) = \det \nabla u = r^{1-n} f^{n-1}\dot{f}$ and suppressed the arguments of F, namely $\mathrm{F} = \mathrm{F}(r^2, f^2)$. To simplify (5.3.7) further we note that,

$$
\partial_2 \mathrm{F} L(\nabla u)^t u = \frac{1}{2}\frac{d\mathrm{F}}{dr}L - r\partial_1 \mathrm{F} L\theta, \tag{5.3.8}
$$

where above for ease of notation we have suppressed the arguments of L, i.e. $L = L(f, \sqrt{n}|\dot{\mathrm{Q}}\theta|)$. Therefore from (5.3.6) and (5.3.8) we obtain that,

$$
\begin{aligned}
& \partial_2 \mathrm{F} L(\nabla u)^t u - \frac{d\mathrm{F}}{dr}(\nabla u)^t \nabla u\, \theta = \\
& \qquad = -\frac{d\mathrm{F}}{dr}\left(\frac{1}{2}L + \frac{f^2}{r^2}\left[r\mathrm{Q}^t\dot{\mathrm{Q}} - (n-1)\mathbf{I}_n\right]\right)\theta - r\partial_1 \mathrm{F} L\theta.
\end{aligned}
\tag{5.3.9}
$$

Hence from the above it is apparent that (5.3.7) becomes,

$$
\begin{aligned}
-\mathrm{F}(\nabla u)^t \Delta u = & \frac{d\mathrm{F}}{dr}\left(\frac{1}{2}L + \frac{f^2}{r^2}\left[r\mathrm{Q}^t\dot{\mathrm{Q}} - (n-1)\mathbf{I}_n\right]\right)\theta \\
& + r\partial_1 \mathrm{F} L\theta + \frac{\gamma}{\varepsilon}\frac{d}{dr}\left[\dot{h}(\gamma)\right]\theta.
\end{aligned}
\tag{5.3.10}
$$

However from the reduced Euler-Lagrange system (5.2.7) we know that,

$$\frac{d}{dr}\left[r^{n-1}\mathrm{F}(r^2,f^2)f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right]=0, \tag{5.3.11}$$

which by expanding out the derivative gives the following useful identity,

$$\frac{f^2}{r}\mathrm{Q}^t\dot{\mathrm{Q}}\frac{d\mathrm{F}}{dr}=-\frac{\mathrm{F}}{r^n}\frac{d}{dr}\left[r^{n-1}f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right]. \tag{5.3.12}$$

This identity when used in (5.3.10) simplifies the expression to the following,

$$\begin{aligned}\mathrm{F}&\left[\frac{1}{r^n}\frac{d}{dr}\left[r^{n-1}f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right]\theta-(\nabla u)^t\Delta u\right]=\\&=\frac{d\mathrm{F}}{dr}\left(\frac{L}{2}-\frac{(n-1)f^2}{r^2}\right)\theta+r\partial_1\mathrm{F}L\theta+\frac{\gamma}{\varepsilon}\frac{d}{dr}\left[\dot{h}\left(\gamma\right)\right]\theta.\end{aligned} \tag{5.3.13}$$

To make any further progress on the simplification of (5.3.13) we must firstly note that,

$$\begin{aligned}\mathrm{F}(\nabla u)^t\Delta u=\mathrm{F}\Bigg\{&\frac{f}{r}\left[\beta\mathrm{Q}^t\dot{\mathrm{Q}}+f\mathrm{Q}^t\ddot{\mathrm{Q}}\right]+f\left(\dot{f}-\frac{f}{r}\right)\langle\mathrm{Q}\theta,\ddot{\mathrm{Q}}\theta\rangle\mathbf{I}_n+\alpha\dot{f}\mathbf{I}_n\\&+\left[\beta|\dot{\mathrm{Q}}\theta|^2+f\langle\dot{\mathrm{Q}}\theta,\ddot{\mathrm{Q}}\theta\rangle\right]f\mathbf{I}_n\Bigg\}\theta.\end{aligned} \tag{5.3.14}$$

Additionally we observe that, (recall that $\beta=2\dot{f}+r^{-1}(n-1)f$)

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right]=f\left[\beta\mathrm{Q}^t\dot{\mathrm{Q}}+f\dot{\mathrm{Q}}^t\dot{\mathrm{Q}}+f\mathrm{Q}^t\ddot{\mathrm{Q}}\right], \tag{5.3.15}$$

which when we multiply both sides by θ and the take the inner product with $\mathrm{Q}^t\dot{\mathrm{Q}}\theta$ gives that,

$$\left\langle\frac{1}{r^{n-1}}\dot{\mathrm{Q}}^t\mathrm{Q}\frac{d}{dr}\left[r^{n-1}f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right]\theta,\theta\right\rangle=f\left[\beta|\dot{\mathrm{Q}}\theta|^2+f\langle\dot{\mathrm{Q}}\theta,\ddot{\mathrm{Q}}\theta\rangle\right]. \tag{5.3.16}$$

With the above in mind we can see that (5.3.14) becomes,

$$\begin{aligned}\mathrm{F}(\nabla u)^t\Delta u=\mathrm{F}\Bigg\{&\frac{f}{r}\left[\beta\mathrm{Q}^t\dot{\mathrm{Q}}+f\mathrm{Q}^t\ddot{\mathrm{Q}}\right]+f\left(\dot{f}-\frac{f}{r}\right)\langle\mathrm{Q}\theta,\ddot{\mathrm{Q}}\theta\rangle\mathbf{I}_n+\alpha\dot{f}\mathbf{I}_n\\&+\left\langle\frac{1}{r^{n-1}}\dot{\mathrm{Q}}^t\mathrm{Q}\frac{d}{dr}\left[r^{n-1}f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right]\theta,\theta\right\rangle\mathbf{I}_n\Bigg\}\theta.\end{aligned} \tag{5.3.17}$$

Therefore using (5.3.12) we find that (5.3.17) can be written as,

$$\begin{aligned}\mathrm{F}(\nabla u)^t\Delta u=\mathrm{F}\Bigg\{&\frac{f}{r}\left[\beta\mathrm{Q}^t\dot{\mathrm{Q}}+f\mathrm{Q}^t\ddot{\mathrm{Q}}\right]\\&+f\left(\dot{f}-\frac{f}{r}\right)\langle\mathrm{Q}\theta,\ddot{\mathrm{Q}}\theta\rangle\mathbf{I}_n+\alpha\dot{f}\mathbf{I}_n\Bigg\}\theta-f^2\frac{d\mathrm{F}}{dr}|\dot{\mathrm{Q}}\theta|^2\theta.\end{aligned} \tag{5.3.18}$$

Moreover combining (5.3.18) with the previously obtained identity (5.3.15) we find that

$$\begin{aligned}\mathrm{F}(\nabla u)^t\Delta u = \mathrm{F}\Bigg\{&\frac{1}{r^n}\frac{d}{dr}\left[r^{n-1}f^2\dot{\mathrm{Q}}^t\dot{\mathrm{Q}}\right] - \frac{f^2}{r}\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} \\ &+ f\left(\dot{f} - \frac{f}{r}\right)\langle \mathrm{Q}\theta, \ddot{\mathrm{Q}}\theta\rangle \mathbf{I}_n + \alpha\dot{f}\mathbf{I}_n\Bigg\}\theta - f^2\frac{d\mathrm{F}}{dr}|\dot{\mathrm{Q}}\theta|^2\theta. \end{aligned} \tag{5.3.19}$$

Furthermore, $|\dot{\mathrm{Q}}\theta|^2 + \langle \mathrm{Q}\theta, \ddot{\mathrm{Q}}\theta\rangle = d\langle \mathrm{Q}\theta, \dot{\mathrm{Q}}\theta\rangle/dr = 0$ which implies that $\langle \mathrm{Q}\theta, \ddot{\mathrm{Q}}\theta\rangle = -|\dot{\mathrm{Q}}\theta|$. Therefore,

$$\begin{aligned}\mathrm{F}(\nabla u)^t\Delta u = \mathrm{F}\Bigg\{&\frac{1}{r^n}\frac{d}{dr}\left[r^{n-1}f^2\dot{\mathrm{Q}}^t\dot{\mathrm{Q}}\right] - \frac{f^2}{r}\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} \\ &- f\left(\dot{f} - \frac{f}{r}\right)|\dot{\mathrm{Q}}\theta|^2 \mathbf{I}_n + \alpha\dot{f}\mathbf{I}_n\Bigg\}\theta - f^2\frac{d\mathrm{F}}{dr}|\dot{\mathrm{Q}}\theta|^2\theta. \end{aligned} \tag{5.3.20}$$

Hence using (5.3.20) in our previous expression for the full Euler-Lagrange system (5.3.13) gives that,

$$\begin{aligned}&\mathrm{F}\left\{\frac{f^2}{r}\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} + f(\dot{f} - \frac{f}{r})|\dot{\mathrm{Q}}\theta|^2\mathbf{I}_n - \alpha\dot{f}\mathbf{I}_n\right\}\theta \\ &\quad = \frac{d\mathrm{F}}{dr}\left(\frac{L}{2} - f^2|\dot{\mathrm{Q}}\theta|^2 - \frac{(n-1)f^2}{r^2}\right)\theta + rL\partial_1\mathrm{F}\theta + \frac{\gamma}{\varepsilon}\frac{d}{dr}\left[\dot{h}\left(\gamma\right)\right]\theta. \end{aligned} \tag{5.3.21}$$

Now recall that the reduce Euler-Lagrange implies that,

$$\frac{\gamma}{\varepsilon}\frac{d}{dr}\left[\dot{h}\left(\gamma\right)\right] = \dot{f}f\left[L\,\partial_2\mathrm{F} + \mathrm{F}\left(\frac{n-1}{r^2} + \frac{|\dot{\mathrm{Q}}|^2}{n}\right)\right] - \frac{\dot{f}}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\mathrm{F}\dot{f}\right]. \tag{5.3.22}$$

Therefore the full Euler-Lagrange system can be expanded further to take the form,

$$\begin{aligned}&\mathrm{F}\left\{\frac{f^2}{r}\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} + f\left(\dot{f} - \frac{f}{r}\right)|\dot{\mathrm{Q}}\theta|^2\mathbf{I}_n - \alpha\dot{f}\mathbf{I}_n\right\}\theta \\ &= \frac{d\mathrm{F}}{dr}\left(\frac{L}{2} - f^2|\dot{\mathrm{Q}}\theta|^2 - \frac{(n-1)f^2}{r^2}\right)\theta + rL\partial_1\mathrm{F}\theta - \frac{\dot{f}}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\mathrm{F}\dot{f}\right]\theta \\ &\qquad + \dot{f}f\left[L\,\partial_2\mathrm{F} + \mathrm{F}\left(\frac{n-1}{r^2} + \frac{|\dot{\mathrm{Q}}|^2}{n}\right)\right]\theta. \end{aligned} \tag{5.3.23}$$

Hence observing that $2r\partial_1\mathrm{F} + 2f\dot{f}\partial_2\mathrm{F} = d\mathrm{F}/dr$ along with other simple manipulations gives

$$\begin{aligned}&\mathrm{F}\left\{\frac{f^2}{r}\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} + f\left(\dot{f} - \frac{f}{r}\right)|\dot{\mathrm{Q}}\theta|^2\mathbf{I}_n - \alpha\dot{f}\mathbf{I}_n\right\}\theta \\ &\quad = \frac{d\mathrm{F}}{dr}\left(L - f^2|\dot{\mathrm{Q}}\theta|^2 - \frac{(n-1)f^2}{r^2} - \dot{f}^2\right)\theta - \mathrm{F}\frac{\dot{f}}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\dot{f}\right]\theta \\ &\qquad + \dot{f}f\,\mathrm{F}\left(\frac{n-1}{r^2} + \frac{|\dot{\mathrm{Q}}|^2}{n}\right)\theta \end{aligned} \tag{5.3.24}$$

Furthermore, since $L = L(\dot{f}, \sqrt{n}|\dot{\mathrm{Q}}\theta|) = f^2|\dot{\mathrm{Q}}\theta|^2 + (n-1)r^{-2}f^2 + \dot{f}^2$ the first term on the right hand side of the above is zero. Additionally,

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\dot{f}\right] = \frac{n-1}{r}\dot{f} + \ddot{f} = \alpha + \frac{(n-1)f}{r^2}, \tag{5.3.25}$$

and therefore we obtain the following expression for the full Euler-Lagrange system,

$$\mathrm{F}\left\{\frac{f^2}{r}\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} + f(\dot{f} - \frac{f}{r})|\dot{\mathrm{Q}}\theta|^2\mathbf{I}_n - \alpha\dot{f}\mathbf{I}_n\right\}\theta = \mathrm{F}\left[\dot{f}f\frac{|\dot{\mathrm{Q}}|^2}{n} - \alpha\dot{f}\right]\theta. \tag{5.3.26}$$

Thus straightforward cancellation and rearrangement gives the following equivalent identity for a twist mapping to solve the full Euler-Lagrange system,

$$0 = \mathrm{F}\left[\frac{f^2}{r}\left(\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} - |\dot{\mathrm{Q}}\theta|\mathbf{I}_n\right) + f\dot{f}\left(|\dot{\mathrm{Q}}\theta|^2 - \frac{|\dot{\mathrm{Q}}|^2}{n}\right)\mathbf{I}_n\right]\theta. \tag{5.3.27}$$

Moreover, as $r, f, \mathrm{F} > 0$ the above implies that,

$$\mathbb{EL}[u] = 0 \iff \left[f\left(\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} - |\dot{\mathrm{Q}}\theta|\mathbf{I}_n\right) + r\dot{f}\left(|\dot{\mathrm{Q}}\theta|^2 - \frac{|\dot{\mathrm{Q}}|^2}{n}\right)\mathbf{I}_n\right]\theta = 0. \tag{5.3.28}$$

A direct calculation verifies in the case

$$\mathrm{Q} = \begin{cases} P\exp(2\pi k\beta\mathrm{J})P^t & n \text{ even}, \\ \mathbf{I}_n & n \text{ odd}, \end{cases} \tag{5.3.29}$$

with β as in (5.3.2) that,

$$\left[f\left(\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} - |\dot{\mathrm{Q}}\theta|\mathbf{I}_n\right) + r\dot{f}\left(|\dot{\mathrm{Q}}\theta|^2 - \frac{|\dot{\mathrm{Q}}|^2}{n}\right)\mathbf{I}_n\right]\theta = 0. \tag{5.3.30}$$

Thus we have proved that,

$$(5.1.8) \Longleftarrow \mathbb{EL}[f, \mathrm{Q}] = 0 \ \& \ \mathrm{Q} = \begin{cases} P\exp(2\pi k\beta\mathrm{J})P^t & n \text{ even}, \\ \mathbf{I}_n & n \text{ odd}, \end{cases}. \tag{5.3.31}$$

For the reverse implication we observe from (5.3.27) that for each $\theta \in \mathbb{S}^{n-1}$ the following relationship between the norm of $|\dot{\mathrm{Q}}|$ and $|\dot{\mathrm{Q}}\theta|$ is satisfied,

$$\begin{aligned} 0 &= \left\langle\left[f\left(\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} - |\dot{\mathrm{Q}}\theta|\mathbf{I}_n\right) + r\dot{f}\left(|\dot{\mathrm{Q}}\theta|^2 - \frac{|\dot{\mathrm{Q}}|^2}{n}\right)\mathbf{I}_n\right]\theta, \theta\right\rangle \\ &= r\dot{f}\left(|\dot{\mathrm{Q}}\theta|^2 - \frac{|\dot{\mathrm{Q}}|^2}{n}\right). \end{aligned} \tag{5.3.32}$$

Hence, as $r > 0$ and $\dot{f} > 0$ we can divide through by r and $\dot{f}$ to obtain,

$$\mathbb{EL}[u] = 0 \Longrightarrow |\dot{\mathrm{Q}}\theta|^2 = \frac{|\dot{\mathrm{Q}}|^2}{n}, \tag{5.3.33}$$

for all $\theta \in \mathbb{S}^{n-1}$. As shown in [66] the right hand side is only possible if all the eigenvalues of $\dot{\mathrm{Q}}^t\dot{\mathrm{Q}}$ are equal. Namely suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\dot{\mathrm{Q}}^t\dot{\mathrm{Q}}$ with eigenvectors given by $\theta_1, \ldots, \theta_n \in \mathbb{S}^{n-1}$. Then (5.3.33) gives that,

$$|\dot{\mathrm{Q}}\theta_j|^2 = \langle \dot{\mathrm{Q}}^t\dot{\mathrm{Q}}\,\theta_j, \theta_j\rangle = \lambda_j = \frac{|\dot{\mathrm{Q}}|^2}{n}, \tag{5.3.34}$$

for $1 \le j \le n$. Therefore when the dimension of the underlying annulus is *odd* we know by the above that all the eigenvalues of $\dot{\mathrm{Q}}^t\dot{\mathrm{Q}}$ are zero as there is always one such eigenvalue which is zero. Thus when n is odd we have that $\dot{\mathrm{Q}} = 0$ and hence,

$$\dot{\mathrm{Q}} = 0 \implies \mathrm{Q}(r) = \mathbf{I}_n, \tag{5.3.35}$$

since we require that $\mathrm{Q}(a) = \mathrm{Q}(b) = \mathbf{I}_n$ which proves the theorem in the case that n is odd. However when n is *even* the condition that $\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} = n^{-1}|\dot{\mathrm{Q}}|^2\mathbf{I}_n$ combined with the fact that $\mathrm{Q}^t\dot{\mathrm{Q}}$ is skew-symmetric implies that $\dot{\mathrm{Q}}(r) \in \mathbb{R}\mathbf{SO}(n)$[1] for each $r \in (a, b)$. To see that this is the case observe that $\mathrm{Q}^t\dot{\mathrm{Q}}$ being skew-symmetric and $\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} = n^{-1}|\dot{\mathrm{Q}}|^2\mathbf{I}_n$ implies that $\exists P \in \mathbf{SO}(n)$ such that $\mathrm{Q}^t\dot{\mathrm{Q}} = n^{-1/2}|\dot{\mathrm{Q}}|P\mathrm{diag}(\mathrm{J}_2, \ldots, \mathrm{J}_2)P^t \in \mathbb{R}\mathbf{SO}(n)$, which in turn implies that $\dot{\mathrm{Q}} \in \mathbb{R}\mathbf{SO}(n)$. Moreover, $\mathbb{EL}[f, \mathrm{Q}] = 0$ gives that $f \in \mathbf{C}^2(a, b)$ solves,

$$\begin{aligned}
&\frac{d}{dr}\left[r^{n-1}\mathrm{F}(r^2, f^2)\dot{f}\right] + \frac{f^{n-1}}{\varepsilon}\frac{d}{dr}\left[\dot{h}\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)\right] \\
&= fr^{n-1}\left[\partial_2\mathrm{F}(r^2, f^2)L(f, \sqrt{c}) + \mathrm{F}(r^2, f^2)\left(\frac{n-1}{r^2} + \frac{c(r)}{n}\right)\right],
\end{aligned} \tag{5.3.36}$$

where $c = c(r) = |\dot{\mathrm{Q}}|^2 \in \mathbf{C}^1(a, b)$. Furthermore, $\mathbb{EL}[f, \mathrm{Q}] = 0$ also gives the following ODE in Q. Namely,

$$\frac{d}{dr}\left[r^{n-1}\mathrm{F}(r^2, f^2)f^2\mathrm{Q}^t\dot{\mathrm{Q}}\right] = 0, \tag{5.3.37}$$

which for any $k \in \mathbb{Z}$ and $P \in \mathbf{SO}(n)$ the rotation matrix $\mathrm{Q}(r) = P\exp(2\pi k\beta(r)\mathrm{J})P^t$ with β solves (5.3.37). It can also be seen that any solution of (5.3.37) satisfying the constraint that $\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} = n^{-1}|\dot{\mathrm{Q}}|^2\mathbf{I}_n$ is of this form. This fact is a consequence of any solution to (5.3.37) satisfying,

$$r^{n-1}\mathrm{F}(r^2, f^2)f^2\mathrm{Q}^t\dot{\mathrm{Q}} = C = P\mathrm{diag}(c_1\mathrm{J}_2, \ldots, c_n\mathrm{J}_2)P^t, \tag{5.3.38}$$

for some skew-symmetric matrix C. Then C being skew-symmetric means that it has the form $P\mathrm{diag}(c_1\mathrm{J}_2, \ldots, c_n\mathrm{J}_2)P^t$ for some $P \in \mathbf{SO}(n)$ and $c_j \in \mathbb{R}$. However the constraint that $\dot{\mathrm{Q}}^t\dot{\mathrm{Q}} = n^{-1}|\dot{\mathrm{Q}}|^2\mathbf{I}_n$ enforces that $c_1 = \cdots = c_n$ and therefore Q must be a solution of,

$$r^{n-1}\mathrm{F}(r^2, f^2)f^2\mathrm{Q}^t\dot{\mathrm{Q}} = cP\mathrm{diag}(\mathrm{J}_2, \ldots, \mathrm{J}_2)P^t, \tag{5.3.39}$$

[1]Note that $\mathbb{R}\mathbf{SO}(n) = \{\mathbf{F} = c\mathrm{Q} : c \in \mathbb{R}, \mathrm{Q} \in \mathbf{SO}(n)\}$.

Then with the matrix $P \in \mathbf{SO}(n)$ and the constant c fixed the ODE (5.3.39) with boundary conditions $\mathrm{Q}(a) = \mathrm{Q}(b) = \mathbf{I}_n$ has a unique solution given above. Hence when n is even we have just proved that,

$$\mathbb{EL}[u] = 0 \Longrightarrow \begin{cases} \mathbb{EL}[f, \mathrm{Q}] = 0, \\ \mathrm{Q} = P\exp(2\pi k\beta \mathrm{J})P^t, \end{cases} \tag{5.3.40}$$

and this therefore completes our proof. □

5.4 Existence and form of twist mappings which are equilibria of $\mathbb{E}_\varepsilon$

The goal of this section is to utilise the equivalence result, Theorem 5.3.1, of the previous section to prove the existence of twist mappings u which are classical solutions to the full Euler-Lagrange system (5.1.8). Moreover, we can in fact prove the existence of countably many twist equilibria of $\mathbb{E}_\varepsilon[u]$ when n is even and the existence of at least one when n is odd. In order to achieve this existence we need to impose the additional asymptotic assumption, mentioned in the introduction, on the convex function h. Namely, we assume that h satisfies **[A5]**. The need for this assumption is technical and necessary in our argument to prove the regularity of radial part f of the twist mappings.

Theorem 5.4.1. (n even) *For each $k \in \mathbb{Z}$ and provided h satisfies* **[A5]** *there exists a $f \in \mathbf{C}^2[a,b]$ such that $\dot{f} > 0$ on $[a,b]$ and*

$$\mathbb{EL}[f, \mathrm{Q}] = 0, \tag{5.4.1}$$

where $\mathrm{Q} = P\exp(2\pi k\beta\mathrm{J})P^t$ and β is given by (5.3.2). *Furthermore, the twist mapping*

$$u(x) = f(r)P\exp(2\pi k\beta\mathrm{J})P^t\theta, \tag{5.4.2}$$

is a classical solution to the full Euler-Lagrange system (5.1.8).

(n odd) *Again provided that h satisfies* **[A5]** *there exists a $f \in \mathbf{C}^2[a,b]$ such that $\dot{f} > 0$ on $[a,b]$ and*

$$\mathbb{EL}[f, \mathbf{I}_n] = 0. \tag{5.4.3}$$

Furthermore, the twist mapping

$$u(x) = f(r)\theta, \tag{5.4.4}$$

is a classical solution to the full Euler-Lagrange system (5.1.8).

Proof. Let us firstly work in the case that n is even and fix any arbitrary $k \in \mathbb{Z}$. To prove the existence of such an $f \in \mathbf{C}^2[a,b]$ we take a variational approach and consider the energy given by,

$$\begin{aligned}\mathscr{E}[(f,\beta);[a,b]] = &\int_a^b r^{n-1}\mathrm{F}(r^2,f^2)\left[(n-1)\frac{f^2}{r^2} + \dot{f}^2 + (2\pi k)^2 f^2\dot{\beta}^2\right] dr \\ &+ \int_a^b \frac{2}{\varepsilon} h\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right) r^{n-1} dr. \end{aligned} \tag{5.4.5}$$

This energy $\mathscr{E}$ is nothing other than the restricted energy $\mathbb{K}_\varepsilon[(f,\mathrm{Q})]$ when Q takes the form $\mathrm{Q}(r) = P\exp(2\pi k\beta(r)\mathrm{J})P^t$ for some function β. Here we consider the energy $\mathscr{E}$ over the space of admissible functions,

$$\mathcal{B} = \{(f,\beta) : f \in \mathscr{T}_+ \text{ and } \beta \in W^{1,2}(a,b) \ s.t.\ \beta(a) = 0 \ \beta(b) = 1\}. \tag{5.4.6}$$

It is relatively straightforward to verify, via the direct method, the existence of $(f,\beta) \in \mathcal{B}$ such that,

$$\mathscr{E}[(f,\beta)] = \inf_{\mathcal{B}} \mathscr{E}[\cdot], \tag{5.4.7}$$

and this is left to the reader to check. Moreover, following the same approach in [66] Theorem 5.3 it can be shown that $f \in \mathbf{C}^2[a,b]$ with $\dot{f} > 0$ on $[a,b]$ and $\beta \in \mathbf{C}^2[a,b]$. To this end let us firstly fix some notation,

$$\begin{aligned}\mathbf{G}_k(r,t,y,p) &= r^{n-1}\left\{\mathrm{F}(r^2,y^2)\left[(n-1)\left(\frac{y}{r}\right)^2 + p^2 + (2\pi k)^2 y^2 t^2\right]\right. \\ &\qquad\qquad\left. + \frac{2}{\varepsilon}h\left(p\left(\frac{y}{r}\right)^{n-1}\right)\right\} \\ &= r^{n-1}\left\{\mathrm{F}(r^2,y^2)L_k(r,t,y,p) + \frac{2}{\varepsilon}h\left(p\left(\frac{y}{r}\right)^{n-1}\right)\right\},\end{aligned} \tag{5.4.8}$$

where above we let $L_k(r,t,y,p) = (n-1)r^{-2}y^2 + p^2 + (2\pi k)^2 y^2 t^2$. With this notation at hand we shall now follow the spirit of [66] and break the proof of this even dimensional case into 3 steps.

(**Step 1:** $\beta \in \mathbf{C}^1$) Firstly as $f \in \mathbf{C}[a,b]$ and $a \le f \le b$ we obtain that $\beta \in \mathbf{C}^1[a,b]$ since it is a minimiser of (also note $0 < c \le \mathrm{F}$)

$$\int_a^b r^{n-1}\mathrm{F}(r^2,f^2) f^2\dot{\beta}^2\, dr, \tag{5.4.9}$$

amongst all β with $(f,\beta) \in \mathcal{B}$.

(**Step 2:** $f \in \mathbf{C}^1$ and $\dot{f} > 0$) To prove that $f \in \mathbf{C}^1[a,b]$ we proceed by following the approach of Ball in [9]. Firstly for a fixed $j \in \mathbb{N}$ we let $A_j = \{r \in (a,b) : j^{-1} \le \dot{f} \le j\}$ and then pick any $\zeta \in L^\infty(a,b)$ such that,

$$\int_{A_j} \zeta \, dr = 0. \tag{5.4.10}$$

Then for $\varrho \in \mathbb{R}$ we define a new radial function f_ϱ to be,

$$f_\varrho(r) := f(r) + \varrho \int_a^r \zeta(s)\chi_{A_j} \, ds. \tag{5.4.11}$$

Now provided that we pick ϱ small enough, i.e. such that $|\varrho|\|\zeta\|_\infty < j^{-1}$, then $f_\varrho \in \mathscr{F}_+$ with $\dot{f}_\varrho = \dot{f}$ for *a.e.* $r \notin A_j$. Furthermore as (f,β) are minimisers of $\mathscr{E}$ over $\mathcal{B}$ we know that,

$$0 = \frac{d}{d\varrho}\mathscr{E}[f_\varrho,\beta]_{|_{\varrho=0}} = \lim_{\varrho\to 0}\int_a^b \frac{\mathbf{G}_k(r,\beta,f_\varrho,\dot{f}_\varrho) - \mathbf{G}_k(r,\beta,f,\dot{f})}{\varrho} \, dr \tag{5.4.12}$$

The goal here is to pass the limit into the integral and therefore our task is to show that this is in fact possible. To this end we shall show that the integrand can be suitably dominated. Firstly, since we are taking ϱ small enough such that $|\varrho|\|\zeta\|_\infty < j^{-1}$ it is straightforward to see that,

$$|f_\varrho| < |f| + j^{-1}(b-a), \qquad |\dot{f}_\varrho| < |\dot{f}| + j^{-1}. \tag{5.4.13}$$

Moreover we have that,

$$\left|\frac{\dot{f}_\varrho^2 - \dot{f}^2}{\varrho}\right| \le \left|\frac{\dot{f}_\varrho - \dot{f}}{\varrho}\right|\left(|\dot{f}_\varrho| + |\dot{f}|\right) \le \|\zeta\|_\infty\left(2|\dot{f}| + j^{-1}\right), \tag{5.4.14}$$

and additionally that,

$$\left|\frac{f_\varrho^2 - f^2}{\varrho}\right| \le \left|\frac{f_\varrho - f}{\varrho}\right|(|f_\varrho| + |f|) \le \|\zeta\|_\infty(b-a)\left(2|f| + j^{-1}(b-a)\right), \tag{5.4.15}$$

Therefore combining (5.4.14), (5.4.15) and (5.4.13) with the fact that $a \le r, f \le b$ we obtain that there are constants $c_1, c_2, c_3 > 0$ such that,

$$\left|\frac{L_k(r,\dot{\beta},f_\varrho,\dot{f}_\varrho) - L_k(r,\dot{\beta},f,\dot{f})}{\varrho}\right| \le c_1 + c_2\dot{\beta}^2 + c_3|\dot{f}| =: H_1(r) \tag{5.4.16}$$

and $H_1 \in L^1(a,b)$. Furthermore a basic application of the triangle inequality and (5.4.16) gives,

$$\begin{aligned}\mathcal{I}_0 = \left|\frac{\mathrm{F}(r^2,f_\varrho^2)L_k(r,\dot{\beta},f_\varrho,\dot{f}_\varrho) - \mathrm{F}(r^2,f^2)L_k(r,\dot{\beta},f,\dot{f})}{\varrho}\right| &\le |\mathrm{F}(r^2,f_\varrho^2)|H_1(r) \\ &+ \left|\frac{\mathrm{F}(r^2,f_\varrho^2) - \mathrm{F}(r^2,f^2)}{\varrho}\right|L_k(r,\dot{\beta},f,\dot{f}).\end{aligned} \tag{5.4.17}$$

Again using (5.4.13) with $\mathrm{F} \in \mathbf{C}^\infty(\mathscr{R})$ we obtain via the mean value theorem that there are constants $c_4, c_5 > 0$ independent of ϱ such that,

$$\mathcal{I}_0 \leq c_4 H_1(r) + c_5 L_k(r, \dot{\beta}, f, \dot{f}) =: H_2(r), \tag{5.4.18}$$

where again it is clear that $H_2(r) \in L^1(a, b)$. Hence in order to show that the integrand in (5.4.12) is dominated we are left to show that,

$$\left| \frac{k(r, f_\varrho, \dot{f}_\varrho) - k(r, f, \dot{f})}{\varrho} \right|, \tag{5.4.19}$$

is suitably bounded where above we have used the notation,

$$k(s_1, s_2, s_3) = h\left(s_3 \left(\frac{s_2}{s_1} \right)^{n-1} \right), \tag{5.4.20}$$

and therefore as $h \in \mathbf{C}^2(0, \infty)$ we obtain that $k \in \mathbf{C}^2$ for $s_1, s_2, s_3 > 0$. Now with this notation set we note that,

$$\begin{aligned} \left| \frac{k(r, f_\varrho, \dot{f}_\varrho) - k(r, f, \dot{f})}{\varrho} \right| &\leq \left| \frac{k(r, f_\varrho, \dot{f}_\varrho) - k(r, f_\varrho, \dot{f})}{\varrho} \right| + \left| \frac{k(r, f_\varrho, \dot{f}) - k(r, f, \dot{f})}{\varrho} \right| \\ &= \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{5.4.21}$$

Note that for *a.e.* $r \notin A_j$ the term $\mathcal{I}_1 = 0$ and when $r \in A_j$ we know by an application of the mean value theorem that $\exists c > 0$ independent of ϱ such that $\mathcal{I}_1 \leq c$. Hence we are just left to show that $\mathcal{I}_2$ is suitably bounded. In order to do this we again apply the mean value theorem to obtain that,

$$\left| \frac{k(r, f_\varrho, \dot{f}) - k(r, f, \dot{f})}{\varrho} \right| = \left| \partial_2 k(r, f + \delta[f_\varrho - f], \dot{f}) \right| \left| \frac{f_\varrho - f}{\varrho} \right|, \tag{5.4.22}$$

where $\delta(\varrho, r) \in [0, 1]$. Moreover a simple manipulation gives that $f + \delta[f_\varrho - f] = \alpha f$ where,

$$\alpha = 1 + \delta \varrho \frac{1}{f} \int_a^r \zeta \chi_{A_j} \, ds. \tag{5.4.23}$$

Now recall that from (5.1.4), in the introduction, we know that there exists $\xi > 0$ and $\eta > 0$ such that for all $t \in (0, \infty)$ and $\alpha > 0$ satisfying $|\alpha - 1| < \eta$ we have that,

$$|\dot{h}(\alpha t) t| \leq \xi [h(t) + 1]. \tag{5.4.24}$$

This translates to the following: provided $\alpha > 0$ is such that $|\alpha^{n-1} - 1| < \eta$ then,

$$|s_2 \partial_2 k(s_1, \alpha s_2, s_3)| \leq (n-1)\xi [k(s_1, s_2, s_3) + 1]. \tag{5.4.25}$$

Thus provided ϱ small enough such that α, given by (5.4.23), satisfies the inequality $|\alpha^{n-1} - 1| < \eta$. Then,

$$
\begin{aligned}
\left|\frac{k(r, f_\varrho, \dot{f}) - k(r, f, \dot{f})}{\varrho}\right| &= |\partial_2 k(r, \alpha f, \dot{f})| \left|\frac{f_\varrho - f}{\varrho}\right| \\
&\leq a^{-1} |f \partial_2 k(r, \alpha f, \dot{f})| \left| \int_a^r \zeta \chi_{A_j} \, ds \right| \\
&\leq C[k(r, f, \dot{f}) + 1] \left| \int_a^r \zeta \chi_{A_j} \, ds \right| =: H_3(r), \qquad (5.4.26)
\end{aligned}
$$

where $H_3 \in L^1(a, b)$ and therefore we have shown that the integrand in (5.4.12) is suitably dominated. Thus applying the Dominated convergence theorem in (5.4.12) we obtain that,

$$
\begin{aligned}
0 &= \int_a^b \left[\partial_p \mathbf{G}_k(r, \beta, f, \dot{f}) \zeta \chi_{A_j} + \partial_y \mathbf{G}(r, \beta, f, \dot{f}) \int_a^r \zeta \chi_{A_j} \, ds \right] dr \\
&= \int_a^b \left[\partial_p \mathbf{G}_k(r, \beta, f, \dot{f}) - \int_a^r \partial_y \mathbf{G}(s, \beta, f, \dot{f}) \, ds \right] \zeta \chi_{A_j} \, dr. \qquad (5.4.27)
\end{aligned}
$$

A consequence of $f \in [a, b]$ and $k(s_1, s_2, s_3)$ defined by (5.4.20) satisfying (5.4.25) combined with $\mathscr{E}[f, \beta] < \infty$ is that $\partial_y \mathbf{G}(s, \beta, f, \dot{f})$ is summable on (a, b). Furthermore, ζ satisfies (5.4.10) so we can conclude that,

$$
c_j = \partial_p \mathbf{G}_k(r, \beta, f, \dot{f}) - \int_a^r \partial_y \mathbf{G}(s, \beta, f, \dot{f}) \, ds, \qquad (5.4.28)
$$

for *a.e.* $r \in A_j$ and some constant $c_j \in \mathbb{R}$. Moreover, since the sets A_j are monotone increasing and $\bigcup_{j \in \mathbb{N}} A_j$ is of full measure in (a, b) we conclude that $c_j = c$ is independent of j and,

$$
\partial_p \mathbf{G}_k(r, \beta, f, \dot{f}) = c + \int_a^r \partial_y \mathbf{G}_k(s, \beta, f, \dot{f}) \, ds, \qquad (5.4.29)
$$

for *a.e.* $r \in (a, b)$. Note that the quantity on the right of (5.4.29) is absolutely continuous on $[a, b]$. Furthermore, as the function h in $\mathbf{G}_k$ is convex and satisfies **[A5]** we know that,

$$
\lim_{p \searrow 0} \partial_p \mathbf{G}_k = \lim_{p \searrow 0} \left[2r^{n-1} \mathrm{F}(r^2, y^2) p + \frac{2}{\varepsilon} \left(\frac{f}{r}\right)^{n-1} \dot{h} \left(p \left(\frac{f}{r}\right)^{n-1} \right) \right] = -\infty
$$

and $\lim_{p \nearrow \infty} \partial_p \mathbf{G}_k = \infty$. Thus it is possible to modify $\dot{f}$ on a set of measure zero in (a, b) such that $\dot{f} > 0$ and (5.4.29) holds everywhere in $[a, b]$. This in turn gives that $\partial_p \mathbf{G}_k(r, \dot{\beta}, f, \dot{f})$ is continuous on $[a, b]$ and using classical arguments as in [19] page 57-61, in particular Theorem 2.6(ii), we obtain the continuity of $\dot{f}$ in $[a, b]$. It is important to note that the assumption of $\mathbf{G}$ being $\mathbf{C}^1$ can be replaced by $\partial_p \mathbf{G}$ being continuous and the result of in Theorem 2.6(ii) will still hold.

(**Step 3:** $\beta, f \in \mathbf{C}^2$ and connection to $\mathbb{EL}[f, \mathrm{Q}] = 0$) The fact that $\beta \in \mathbf{C}^2[a, b]$ is then a consequence of $f \in \mathbf{C}^1[a,b]$ and repeating the earlier step for β, whilst the conclusion that $f \in \mathbf{C}^2[a,b]$ follows from $f \in \mathbf{C}^1[a,b]$ and the Hilbert-Weierstrass differentiability theorem *see* [19] Theorem 2.6(iii). Note that the result of Theorem 2.6(iii) also holds if the assumption that $\mathbf{G}$ being $\mathbf{C}^2$ is replaced by $\partial_p \mathbf{G}$ being of class $\mathbf{C}^1$. Therefore $(f,\beta) \in \mathbf{C}^2[a,b] \times \mathbf{C}^2[a,b]$ are classical solutions to the Euler-Lagrange system associated to the energy $\mathscr{E}$, which is given by,

$$\begin{cases} \dfrac{d}{dr}\left[r^{n-1}\mathrm{F}(r^2,f^2)f^2\dot{\beta}\right] = 0, \\ \\ \dfrac{d}{dr}\left[r^{n-1}\mathrm{F}\dot{f}\right] + \dfrac{f^{n-1}}{\varepsilon}\dfrac{d}{dr}(\dot{h}) = fr^{n-1}\left[\partial_2 \mathrm{F} L_k + \mathrm{F}\left(\dfrac{n-1}{r^2} + (2\pi k)^2\dot{\beta}^2\right)\right], \end{cases}$$

where above $L_k = (n-1)r^{-2}f^2 + \dot{f}^2 + (2\pi k)^2 f^2 \dot{\beta}^2$ and $\mathrm{F} = \mathrm{F}(r^2,f^2)$. Therefore, $\beta(r)$ is given by (5.3.2) and correspondingly $\mathbb{EL}[f,\mathrm{Q}] = 0$. Thus, applying Theorem 5.3.1 we have that the corresponding twist mapping is a classical solution to the full Euler-Lagrange system (5.1.8) which completes the proof in the case that n is even.

In the case that n is odd we follow a similar approach and consider the energy when $k=0$ i.e.

$$\begin{aligned} \mathscr{E}[f;[a,b]] = &\int_a^b r^{n-1}\mathrm{F}(r^2,f^2)\left[(n-1)\left(\frac{f}{r}\right)^2 + \dot{f}^2 + nf^2\right]dr \\ &+ \int_a^b \frac{2}{\varepsilon}h\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right)r^{n-1}dr. \end{aligned} \tag{5.4.30}$$

over the space of admissible functions given by $\mathscr{T}_+$. Then as above the minimiser f is attained in $\mathscr{T}_+$ and $f \in \mathscr{T}_+ \cap \mathbf{C}^2[a,b]$. Hence, f is a classical solution to the Euler-Lagrange associated with $\mathscr{E}$, which corresponds to $\mathbb{EL}[f,\mathbf{I}_n] = 0$. Thus, applying Theorem 5.3.1 as before completes our proof. □

5.5 Existence and convergence of L^1 local minimisers in the planar setting

In this section we devote our attention to the case when $n=2$, i.e. the underlying annulus $\mathbf{X} \subset \mathbb{R}^2$. Here the existence of countably many L^1-local minimisers to the energies $\mathbb{E}_\varepsilon[u]$ for each $\varepsilon > 0$ is known. Before giving this existence result we first fix some notation and definitions useful for the understanding of this result. Firstly, any mapping $u \in \mathcal{A}^+(\mathbf{X})$ has a continuous representative which is again denoted by $u \in \mathscr{C}(\mathbf{X})$ where,

$$\mathscr{C}(\mathbf{X}) = \{u \in \mathbf{C}(\mathbf{X},\mathbb{R}^2) : u(x) = x \text{ on } \partial\mathbf{X}\}. \tag{5.5.1}$$

Furthermore it is well known from [74],[75] or [76] that $\mathscr{C}(\mathbf{X})$ can be partitioned into pairwise disjoint sets in the following way,

$$\mathscr{C}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{C}_k(\mathbf{X}), \tag{5.5.2}$$

where each $\mathscr{C}_k(\mathbf{X}) = \{u \in \mathscr{C}(\mathbf{X}) :]u[= k\}$. Here $]u[$ denotes the winding number of the curve $\gamma_\vartheta(r) : u/|u|(r,\vartheta)$ which is written in polar co-ordinates where $\vartheta \in [0, 2\pi]$ is fixed. Note that $]u[$ is independent of the choice of ϑ by the continuity of u. For details on this partitioning and in particular the indexing of the sets $\mathscr{C}(\mathbf{X})$ by $]u[$ the reader is referred to [74] and [76]. The importance of this is that we can partition $\mathcal{A}^+(\mathbf{X})$ as,

$$\mathcal{A}^+(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{A}_k^+, \tag{5.5.3}$$

with $\mathscr{A}_k^+ = \{u \in \mathcal{A}^+(\mathbf{X}) : u \in \mathscr{C}_k(\mathbf{X})\}$ where $u \in \mathscr{C}_k(\mathbf{X})$ we meant in terms of the continuous representative of $u \in \mathcal{A}^+(\mathbf{X})$. For each $k \in \mathbb{Z}$ we shall refer to the set $\mathscr{A}_k^+$ as a homotopy class of $\mathcal{A}^+(\mathbf{X})$. (We use this terminology as the continuous representatives $u, v \in \mathscr{A}_k$ are homotopic. For more on this the interested reader is referred to [74] or [76]). We are now in a position to state the existence result of countably many L^1-local minimisers (*see* [74]).

Theorem 5.5.1. *For each index $k \in \mathbb{Z}$ there exists a $u^k \in \mathscr{A}_k^+$ such that,*

$$\mathbb{E}_\varepsilon[u^k] = \inf_{u\in\mathscr{A}_k} \mathbb{E}_\varepsilon[u]. \tag{5.5.4}$$

Moreover each $u^k \in \mathscr{A}_k$ is a L^1-local minimiser of $\mathbb{E}_\varepsilon$ in $\mathcal{A}^+(\mathbf{X})$.

Proof. This result is essentially a consequence of $\mathbb{E}_\varepsilon$ being polyconvex, $\mathscr{A}_k$ being sequentially weakly closed (for a proof of this result the reader is referred to [24] or [74]) and the following weak continuity result for $\det \nabla u$. Namely, (*see* Proposition 18.28 on page 776 in [78]) if $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\{v_m\} \subset W^{1,n}(\Omega, \mathbb{R}^n)$ satisfies $\det \nabla v_m \geq 0$ *a.e.* in Ω with $v_m \rightharpoonup v$ in $W^{1,n}(\Omega, \mathbb{R}^n)$ as $m \to \infty$. Then upon passing to a subsequence (if needed),

$$\det \nabla v_m \rightharpoonup \det \nabla v, \tag{5.5.5}$$

in $L^1(E)$ for any $E \subset\subset \Omega$. Thus, if $\{v_m\} \subset \mathcal{A}^+(\mathbf{X})$ and $v_m \rightharpoonup v$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ as $m \to \infty$ then by extending each mapping by identity onto a large annulus $\tilde{\mathbf{X}}$ we trivially obtain that $\{\tilde{v}_m\} \subset \mathcal{A}^+(\tilde{\mathbf{X}})$ and $\tilde{v}_m \rightharpoonup \tilde{v}$ in $W^{1,2}(\tilde{\mathbf{X}}, \mathbb{R}^2)$ as $m \to \infty$. Then, by the aforementioned result, upon passing to a subsequence

$$\det \nabla v_m \rightharpoonup \det \nabla v, \tag{5.5.6}$$

in $L^1(\mathbf{X})$. The result is now standard. $\square$

With this existence result at hand we now want to look at what happens to the minimisers in each homotopy class $\mathscr{A}_k^+$ as $\varepsilon \searrow 0$, i.e. as we tend to the limit problem $\varepsilon = 0$. In order to look at this limit problem let us first gather some relevant definitions and results. Firstly we shall define,

$$\mathcal{A}(\mathbf{X}) = \{x \in \mathcal{A}^+(\mathbf{X}) : \det \nabla u = 1 \;\; a.e. \text{ in } \mathbf{X}\}, \tag{5.5.7}$$

which also has its own corresponding homotopy classes given by $\mathscr{A}_k := \mathscr{A}_k^+ \cap \mathcal{A}(\mathbf{X})$ for each $k \in \mathbb{Z}$. Each of these $\mathscr{A}_k$ can be shown to be weakly closed with respect to $W^{1,2}$, *see* [76], and moreover $\mathscr{A}_k \subset \mathscr{A}_k^+$. Furthermore let us denote Φ_ε by,

$$\Phi_\varepsilon(x, y, \mathbf{F}) = \frac{1}{2}\mathrm{F}(|x|^2, |y|^2)|\mathbf{F}|^2 + \frac{1}{\varepsilon}\dot{h}(\det \mathbf{F}) \tag{5.5.8}$$

with the limit as we send $\varepsilon \searrow 0$ being denoted by,

$$\Phi(x, y, \mathbf{F}) = \lim_{\varepsilon \to 0} \Phi_\varepsilon(x, y, \mathbf{F}). \tag{5.5.9}$$

Note that for $u \in \mathcal{A}(\mathbf{X})$ we have that $\Phi(x, u, \nabla u) = 2^{-1}\mathrm{F}(|x|^2, |u|^2)|\nabla u|^2$ *a.e.* in $\mathbf{X}$, whilst for $u \notin \mathcal{A}(\mathbf{X})$ we have $\Phi(x, u, \nabla u) = \infty$ *a.e.* in $\mathbf{X}$. Now we have this notation lets define the limit energy (later in this section we see that $\mathbb{F}$ is in fact the Γ-limit of the energies $\mathbb{E}_\varepsilon$) $\mathbb{F}$ to be

$$\mathbb{F}[u; \mathbf{X}] = \int_{\mathbf{X}} \Phi(x, u, \nabla u)\, dx. \tag{5.5.10}$$

Therefore as $\mathbb{F}$ is only finite on $\mathcal{A}(\mathbf{X})$ we shall consider the limit problem to be minimising,

$$\mathbb{F}[u; \mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} \mathrm{F}(|x|^2, |u|^2)|\nabla u|^2\, dx. \tag{5.5.11}$$

over the respective homotopy classes $\mathscr{A}_k$ for $k \in \mathbb{Z}$. Moreover, we shall make the following restriction on the functions F allowed in the energies $\mathbb{E}_\varepsilon$ and $\mathbb{F}$. Namely that $\mathrm{F} \in \mathbf{C}^\infty(\mathscr{R})$ are of the form,

$$\mathrm{F}(x, y) = \varphi(x)y^{-1}, \tag{5.5.12}$$

where $0 < \varphi \in \mathbf{C}^\infty(0, \infty)$ is such that

$$\frac{d^2}{dt^2}\left[\varphi^{1/2}(t^2)\right] \geq -\frac{d}{dt}\left[\frac{\varphi^{1/2}(t^2)}{t}\right], \tag{5.5.13}$$

for $t \in (0, \infty)$. One example of such a φ is $\varphi(t) = t^\alpha$ with $\alpha^2 \geq 1$. The following result about the minimisers in homotopy classes for the limit problem was proved by the authors in [60].

Theorem 5.5.2. *Let $\mathbb{F}$ denote the limit energy given by,*

$$\mathbb{F}[u;\mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} |u|^{-2}\varphi(|x|^2)|\nabla u|^2\,dx. \tag{5.5.14}$$

where $0 < \varphi \in \mathbf{C}^\infty(0,\infty)$ satisfies (5.5.13). Then for any $k \in \mathbb{Z}$ the energy $\mathbb{F}$ has a unique twist minimiser u^k in the homotopy class $\mathscr{A}_k$. Moreover the unique minimiser is given by,

$$u^k(x) = r\exp(2\pi k\beta)\theta, \tag{5.5.15}$$

with β given as in (5.3.2) for this corresponding setting ($f = r$).

Proof. For a proof the reader is referred to the paper [60]. $\square$

We are now in a position to prove that the minimisers in the homotopy classes $\mathscr{A}_k^+$ converge strongly in $W^{1,2}$ to the unique minimisers u^k of $\mathbb{F}$ over $\mathscr{A}_k$.

Proposition 5.5.3. *Fix a $k \in \mathbb{Z}$ and let u_ε^k denote the minimiser of $\mathbb{E}_\varepsilon$ in the homotopy class $\mathscr{A}_k^+$. Then as $\varepsilon \to 0$ we have that $u_\varepsilon^k \to u^k$ in $W^{1,2}$ where u^k is the unique minimiser (twist) of the general distortion energy $\mathbb{F}$ over $\mathscr{A}_k$.*

Proof. Firstly since we know that $\mathscr{A}_k \subset \mathscr{A}_k^+$ it follows that,

$$\mathbb{E}_\varepsilon[u_\varepsilon^k] \le \mathbb{F}[u^k] \le c\int_{\mathbf{X}} |\nabla u^k|^2\,dx < \infty, \tag{5.5.16}$$

where $0 < c = (2a^2)^{-1}\max_{a\le|x|\le b}\phi(|x|^2)$. Hence the sequence of minimisers $\{u_\varepsilon^k\} \subset W^{1,2}(\mathbf{X},\mathbb{R}^2)$ is bounded in $W^{1,2}$ and therefore the exists a subsequence such that $u_\varepsilon^k \rightharpoonup u \in W^{1,2}(\mathbf{X},\mathbb{R}^2)$ as $\varepsilon \to 0$. However by (5.5.16) we also know that,

$$0 \le \int_{\mathbf{X}} h(\det\nabla u_\varepsilon^k)\,dx \le \varepsilon c\int_{\mathbf{X}} |\nabla u^k|^2\,dx. \tag{5.5.17}$$

Hence as $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0}\int_{\mathbf{X}} h(\det\nabla u_\varepsilon^k)\,dx = 0. \tag{5.5.18}$$

Moreover since $u_\varepsilon^k \rightharpoonup u$ in $W^{1,2}(\mathbf{X},\mathbb{R}^2)$ we know (as in the proof of Theorem 5.5.1) that $\det\nabla u_\varepsilon^k \rightharpoonup \det\nabla u$ in $L^1(\mathbf{X})$. Furthermore, by the assumed convexity of h we have that,

$$0 \le \int_{\mathbf{X}} h(\det\nabla u)\,dx \le \liminf_{\varepsilon\to 0}\int_{\mathbf{X}} h(\det\nabla u_\varepsilon^k)\,dx = 0, \tag{5.5.19}$$

and therefore since $h(\cdot) \ge 0$ with $h(t) = 0$ if and only if $t = 1$ we obtain that,

$$\det\nabla u = 1 \text{ } a.e. \text{ in } \mathbf{X}. \tag{5.5.20}$$

Additionally $\mathscr{A}_k^+$ is weakly closed with respect to $W^{1,2}$ which implies that $u \in \mathscr{A}_k^+$ and, by (5.5.20), $u \in \mathscr{A}_k$. Finally the sequentially weak lower semi-continuity of (5.5.14) with respect to $W^{1,2}$ gives that $\mathbb{F}[u] = \mathbb{F}[u^k]$, which by the uniqueness of $u^k \in \mathscr{A}_k$ as a minimiser of $\mathbb{F}$ we conclude that $u = u^k$. To conclude the strong convergence in $W^{1,2}$ we must first note that by (5.5.16) we have that,

$$\mathbb{F}[u^k] \leq \liminf_{\varepsilon \to 0} \frac{1}{2} \int_{\mathbf{X}} |u^k_\varepsilon|^{-2} \varphi(|x|^2) |\nabla u^k_\varepsilon|^2 \, dx \leq \lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon[u^k_\varepsilon] \leq \mathbb{F}[u^k]. \tag{5.5.21}$$

Thus we additionally have that

$$\lim_{\varepsilon \to 0} \int_{\mathbf{X}} |u^k_\varepsilon|^{-2} \varphi(|x|^2) |\nabla u^k_\varepsilon|^2 \, dx = \int_{\mathbf{X}} |u^k|^{-2} \varphi(|x|^2) |\nabla u^k|^2 \, dx. \tag{5.5.22}$$

Furthermore as $\det \nabla u^k_\varepsilon > 0$ *a.e.* it is possible to pass to a subsequence such that $u^k_\varepsilon \to u^k$ uniformly on $\overline{\mathbf{X}}$. Let us briefly now justify this claim of uniform convergence on a subsequence: firstly $\det \nabla u^k_\varepsilon > 0$ *a.e.* in $\mathbf{X}$ implies that the sequence $\{u^k_\varepsilon\}$ is equi-continuous and uniformly bounded in $\mathscr{C}(\mathbf{X})$. Hence an application of the Arzela-Ascoli's compactness theorem allows us to pass to a subsequence and obtain that $u^k_\varepsilon \to u^k$ in $\mathscr{C}(\mathbf{X})$. Therefore,

$$\left| \int_{\mathbf{X}} \left[|u^k_\varepsilon|^{-2} - |u^k|^{-2} \right] \varphi(|x|^2) |\nabla u^k_\varepsilon|^2 \, dx \right| \leq \sup_{\overline{\mathbf{X}}} \left| \frac{|u^k_\varepsilon|^2 - |u^k|^2}{|u^k_\varepsilon|^2 |u^k|^2} \right| \int_{\mathbf{X}} |\nabla u^k_\varepsilon|^2 \, dx.$$

which we can make arbitrarily small by send $\varepsilon \to 0$. A consequence of this is that,

$$\begin{aligned} \left| \left| \int_{\mathbf{X}} \varphi(|x|^2) \left[\frac{|\nabla u^k_\varepsilon|^2}{|u^k_\varepsilon|^2} - \frac{|\nabla u^k|^2}{|u^k|^2} \right] dx \right| - \left| \int_{\mathbf{X}} \frac{\varphi(|x|^2)}{|u^k|^2} \left[|\nabla u^k_\varepsilon|^2 - |\nabla u^k|^2 \right] dx \right| \right| \\ \leq \left| \int_{\mathbf{X}} \left[|u^k_\varepsilon|^{-2} - |u^k|^{-2} \right] \varphi(|x|^2) |\nabla u^k_\varepsilon|^2 \, dx \right| \to 0, \end{aligned} \tag{5.5.23}$$

as $\varepsilon \to 0$. Hence we obtain by the above and (5.5.22) that,

$$0 = \lim_{\varepsilon \to 0} \left| \int_{\mathbf{X}} \frac{\varphi(|x|^2)}{|u^k|^2} \left[|\nabla u^k_\varepsilon|^2 - |\nabla u^k|^2 \right] dx \right|. \tag{5.5.24}$$

which gives that (below we have denoted $\xi = \varphi(|x|^2)^{1/2}/|u^k| \in \mathbf{C}(\overline{\mathbf{X}})$)

$$\lim_{\varepsilon \to 0} \| \xi \, \nabla u^k_\varepsilon \|_2^2 = \| \xi \, \nabla u^k \|_2^2. \tag{5.5.25}$$

Then noting that $\exists c > 0$ such that $c \leq \xi^2(x)$ for all $x \in \overline{\mathbf{X}}$ and applying the following standard trick we obtain that,

$$c \lim_{\varepsilon \to 0} \| \nabla u^k_\varepsilon - \nabla u^k \|_2^2 \leq \lim_{\varepsilon \to 0} \| \xi \, \nabla u^k_\varepsilon - \xi \, \nabla u^k \|_2^2 \tag{5.5.26}$$

$$= \| \xi \, \nabla u^k \|_2^2 + \lim_{\varepsilon \to 0} \left[\| \xi \, \nabla u^k_\varepsilon \|_2^2 - 2 \langle \nabla u^k_\varepsilon, \xi^2 \, \nabla u^k \rangle \right]$$

$$= 2 \| \xi \, \nabla u^k \|_2^2 - 2 \lim_{\varepsilon \to 0} \langle \nabla u^k_\varepsilon, \xi^2 \, \nabla u^k \rangle = 0, \tag{5.5.27}$$

and therefore we have obtain the strong convergence on the subsequence in $W^{1,2}$. Note that the final equality above comes from $\nabla u^k_\varepsilon \rightharpoonup \nabla u^k$ in L^2 and $\xi \, \nabla u^k \in L^2$. Furthermore, as the limit u^k is unique we obtain that u^k_ε converges strongly to u^k in $W^{1,2}$ on the full sequence. To see this is the case we suppose the contrary then one can subtract a subsequence which doesn't converge to u_k. However this subsequence itself is bounded in $W^{1,2}$ and therefore by similar argument to before we can extract another subsequence which converges to some $u \in \mathscr{A}_k$. Thus we have arrived at our contradiction as by (5.5.21) we will have that $\mathbb{F}[u] = \mathbb{F}[u^k]$. $\square$

Let us now prove the analogue of the results above in the setting of the reduced energy for twist mappings. To this end we shall use the notation of [66] and let $]\mathrm{Q}[$ denote an element of the fundamental group $\pi_1[\mathbf{SO}(n)]$. It is well known that $\pi_1[\mathbf{SO}(n)] \cong \mathbb{Z}$ when $n = 2$ and therefore using this we partition our space of twists $\mathscr{W}$ into $\mathbb{Z}$ components in the following way,

$$\mathscr{W} = \bigcup_{k \in \mathbb{Z}} \mathscr{W}_k, \tag{5.5.28}$$

where each $\mathscr{W}_k = \{(f, \mathrm{Q}) \in \mathscr{W} :]Q[= k\}$ are pairwise disjoint. Moreover, since $0 < c \leq \mathrm{F}(r^2, f^2)$ for all $(f, \mathrm{Q}) \in \mathscr{W}$ we know that $\exists d(n, a, b) > 0$ such that,

$$\mathbb{K}_\varepsilon[f, \mathrm{Q}] \geq d \left(\|\mathrm{Q}\|^2_{W^{1,2}} + \|f\|^2_{W^{1,2}} \right), \tag{5.5.29}$$

for all $(f, \mathrm{Q}) \in \mathscr{W}$. With the coercivity of the reduced energy $\mathbb{K}_\varepsilon$ at hand we can now state the following existence result for minimising stationary loops $(f_k, \mathrm{Q}_k) \in \mathscr{W}_k$. Namely,

Theorem 5.5.4. *For each $k \in \mathbb{Z}$ and $\varepsilon > 0$ there exists a pair $(f^k_\varepsilon, \mathrm{Q}^k_\varepsilon) \in \mathscr{W}_k$ such that,*

$$\mathbb{K}_\varepsilon[f^k_\varepsilon, \mathrm{Q}^k_\varepsilon] = \inf_{\mathscr{W}_k} \mathbb{K}_\varepsilon[f, \mathrm{Q}]. \tag{5.5.30}$$

Proof. The proof follows exactly as that of Theorem 4.2 in [66] given the above coercivity of the reduced energy $\mathbb{K}_\varepsilon$. $\square$

Remark 5.5.1. Before we continue any further we would like to point out that when $n = 2$ any pair $(f, \mathrm{Q}) \in \mathscr{W}_k$ can be represented by $(f, \beta) \in \mathcal{B}$ where the correspondence is $\mathrm{Q}(r) = \exp(2\pi k \beta)$ and the restricted energy is,

$$\mathbb{K}_\varepsilon[f, \mathrm{Q}] = \mathscr{E}^k_\varepsilon[(f, \beta)]. \tag{5.5.31}$$

Therefore, by assuming that h satisfies **[A5]** Theorem 5.4.1 gives for each $k \in \mathbb{Z}$ the existence of the pair $(f^k_\varepsilon, \mathrm{Q}^k_\varepsilon)$ proved in the theorem above. Moreover by Theorem 5.4.1 we know that in this case the pair correspond to a twist u which is a classical solution to the full Euler-Lagrange system, i.e. $\mathbb{EL}[u] = 0$.

Let us now use this existence of minimising stationary loops $(f_k, \mathrm{Q}_k) \in \mathscr{W}_k$ to prove the convergence of these stationary loops to u^k, i.e. the unique minimiser of $\mathbb{F}$ in $\mathscr{A}_k$, which was shown in [60] to be a twist mapping. This result is the corresponding version of Proposition 5.5.3 for the setting of twist mappings.

Proposition 5.5.5. *Fix a $k \in \mathbb{Z}$ and let $(f_\varepsilon^k, \mathrm{Q}_\varepsilon^k)$ denote the minimiser of* (5.2.4) *in the class $\mathscr{W}_k$. Then $(f_\varepsilon^k, \mathrm{Q}_\varepsilon^k) \to (r, \mathrm{Q}_k)$ in $W^{1,2}$ where $\mathrm{Q}_k = \exp(2\pi k \beta)$ and $u^k = r\mathrm{Q}_k[r]\theta$ is the unique minimiser of the energy $\mathbb{F}$.*

Proof. The proof here follows in exactly the same fashion as that of Proposition 5.5.3. Firstly note that $\mathscr{W}_k^1 \subset \mathscr{W}_k$ and therefore,

$$\mathbb{K}_\varepsilon[f_\varepsilon^k, \mathrm{Q}_\varepsilon^k] \leq \mathbb{F}[u^k] \leq C < \infty. \tag{5.5.32}$$

Hence the sequences $\{f_\varepsilon^k\} \subset W^{1,2}(a,b)$ and $\{\mathrm{Q}_\varepsilon^k\} \subset W^{1,2}((a,b), \mathbf{SO}(2))$ are bounded in their respective $W^{1,2}$ space and therefore the exists subsequences such that as $\varepsilon \to 0$,

$$\begin{cases} \mathrm{Q}_\varepsilon^k \to \mathrm{Q} & \text{in } \mathbf{C}([a,b], \mathbf{SO}(2)), \\ \mathrm{Q}_\varepsilon^k \rightharpoonup \mathrm{Q} & \text{in } W^{1,2}((a,b), \mathbf{SO}(2)), \end{cases} \qquad \begin{cases} f_\varepsilon^k \to f & \text{in } \mathbf{C}[a,b], \\ f_\varepsilon^k \rightharpoonup f & \text{in } W^{1,2}(a,b). \end{cases} \tag{5.5.33}$$

Moreover as we know that,

$$0 \leq \int_{\mathbf{X}} h\left(\dot{f}_\varepsilon^k \left(\frac{f_\varepsilon^k}{r} \right)^{n-1} \right) dx \leq \varepsilon c, \tag{5.5.34}$$

we obtain as $\varepsilon \to 0$ that,

$$\lim_{\varepsilon \to 0} \int_{\mathbf{X}} h\left(\dot{f}_\varepsilon^k \left(\frac{f_\varepsilon^k}{r} \right)^{n-1} \right) dx = 0. \tag{5.5.35}$$

Then as $f_\varepsilon^k \to f \in \mathbf{C}[a,b]$ and $f_\varepsilon^k \rightharpoonup f$ in $W^{1,2}$ as $\varepsilon \to 0$ we have that,

$$\dot{f}_\varepsilon^k \left(\frac{f_\varepsilon^k}{r} \right)^{n-1} \rightharpoonup \dot{f} \left(\frac{f}{r} \right)^{n-1}, \tag{5.5.36}$$

in $L^2(a,b)$. Therefore since h is assumed to be convex we have that,

$$0 \leq \int_{\mathbf{X}} h\left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) dx \leq \liminf_{\varepsilon \to 0} \int_{\mathbf{X}} h\left(\dot{f}_\varepsilon^k \left(\frac{f_\varepsilon^k}{r} \right)^{n-1} \right) dx = 0, \tag{5.5.37}$$

which as in the proof of Proposition 5.5.3 implies that,

$$\dot{f} \left(\frac{f}{r} \right)^{n-1} = 1 \text{ } a.e. \text{ in } [a,b]. \tag{5.5.38}$$

This in turn implies that,

$$\frac{d}{dr} f^n = \frac{d}{dr} r^n, \tag{5.5.39}$$

for almost every $r \in [a,b]$ and by the boundary conditions and the continuity of f we obtain that $f(r) = r$ for all $r \in [a,b]$. Hence the pair (r, Q) forms an incompressible twist $u(x) = \mathrm{Q}(r)x$. Furthermore by the uniqueness of twist minimiser to $\mathbb{F}$ we conclude that $\mathrm{Q} = \mathrm{Q}_k$. Finally the strong convergence of $f_\varepsilon^k \to r$ and $\mathrm{Q}_\varepsilon^k \to \mathrm{Q}_k$ in $W^{1,2}$ as $\varepsilon \to 0$ follows from the strong convergence of $u_\varepsilon^k = f_\varepsilon^k \mathrm{Q}_\varepsilon^k \theta \to u^k = r\mathrm{Q}_k\theta$ which is a consequence again of,

$$\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon[u_\varepsilon^k] = \mathbb{F}[u^k]. \tag{5.5.40}$$

as in Proposition 5.5.3. This therefore completes the proof. □

Let us now also briefly consider energies $\mathbb{E}_\varepsilon[u]$ and $\mathbb{F}[u]$ from a Γ-convergence point of view (*see* [25, 26] for an introduction to the subject.). Firstly we extend the respective energies by infinity to $L^1(\mathbf{X}, \mathbb{R}^2)$, i.e. $\mathbb{E}_\varepsilon[u]$ and $\mathbb{F}[u]$ take the value ∞ for any $u \in L^1(\mathbf{X}, \mathbb{R}^2)$ such that $u \notin \mathcal{A}^+(\mathbf{X})$ or $u \notin \mathcal{A}(\mathbf{X})$ respectively. Furthermore recall that

$$\Phi(x, u, \nabla u) = \lim_{\varepsilon \to 0} \Phi_\varepsilon(x, u, \nabla u). \tag{5.5.41}$$

Therefore $\Phi_\varepsilon \nearrow \Phi$ on $\mathbf{X} \times \mathbf{X} \times \mathbb{R}^{n \times n}$ and $\mathbb{E}_\varepsilon \xrightarrow{\Gamma} \mathbb{F}$, namely $\mathbb{F}$ is the Γ-limit of $\mathbb{E}_\varepsilon$ as $\varepsilon \to 0$. Meaning that,

1) For every sequence $\{u_\varepsilon\} \subset L^1(\mathbf{X}, \mathbb{R}^2)$ such that $\|u_\varepsilon - u\|_1 \to 0$ as $\varepsilon \to 0$ for some $u \in L^1(\mathbf{X}, \mathbb{R}^2)$. Then,

$$\mathbb{F}[u] \le \liminf_{\varepsilon \to 0} \mathbb{E}_\varepsilon[u_\varepsilon]. \tag{5.5.42}$$

2) For all $u \in L^1(\mathbf{X}, \mathbb{R}^2)$ there exists a sequence $\{u_\varepsilon\} \subset L^1(\mathbf{X}, \mathbb{R}^2)$ with $\|u_\varepsilon - u\|_1 \to 0$ as $\varepsilon \to 0$ such that,

$$\mathbb{F}[u] \ge \limsup_{\varepsilon \to 0} \mathbb{E}_\varepsilon[u_\varepsilon]. \tag{5.5.43}$$

The above is merely stated for the reader to highlight that throughout this section it is exactly this Γ-convergence of the energies which has been the underlying principle.

5.6 Annular rearrangements and the finer properties of planar mappings in $\mathcal{A}^+(\mathbf{X})$

The purpose of this section is to introduce the reader to some of the particularly useful properties that planar mappings in $\mathcal{A}^+(\mathbf{X})$ possess. Furthermore we shall introduce a particular kind of rearrangement which we call an *annular rearrangement*. This is essentially a specially adapted version of the well known Schwarz rearrangement. Let us begin this section by stating the following lifting result which is valid for mappings $u \in \mathcal{A}^+(\mathbf{X})$. This result will be particularly useful in the next section.

Theorem 5.6.1. *For each* $u \in \mathscr{A}_k^+$ *there exists a corresponding function* $g \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$ *such that* u *has the following lifting,*

$$u(x) = |u|\mathrm{Q}[g]\theta, \tag{5.6.1}$$

and $g(a\omega) = 0, g(b\omega) = 2\pi k$ *for all* $\omega \in \mathbb{S}^1$.

Proof. The proof of this result follows exactly as in [60] Theorem 2.3, where it is proved for the subset of mappings $\mathcal{A}(\mathbf{X})$. □

In addition to this lifting result planar mappings in $\mathcal{A}^+(\mathbf{X})$ have some particularly nice properties which will be useful for us later. Most of these properties will be well known to experts, except perhaps 6), but for the convenience of the reader we formulate them here in a proposition.

Proposition 5.6.2. *Any mapping* $u \in \mathcal{A}^+(\mathbf{X})$ *satisfies the following properties.*

1) u *has a continuous representative, again denoted by* u, *in* $\mathscr{C}(\mathbf{X})$.

2) $u(\overline{\mathbf{X}}) = \overline{\mathbf{X}}$

3) u *is one-to-one almost everywhere in* $\mathbf{X}$.

4) u *satisfies both the* N *and* N^{-1} *property.*

5) The so called (INV) condition holds for u.

Where the (INV) condition means that for all $x_0 \in \mathbf{X}$ *and a.e.* $r \in (0, \mathrm{dist}(x_0, \partial\mathbf{X}))$ *the following two conditions are satisfied,*

$$a)\ u(x) \in im_T(u, \mathbb{B}(x_0, r)) \quad \text{for a.e. } x \in \mathbb{B}(x_0, r), \tag{5.6.2}$$

$$b)\ u(x) \notin im_T(u, \mathbb{B}(x_0, r)) \quad \text{for a.e. } x \in \mathbf{X}\backslash\mathbb{B}(x_0, r). \tag{5.6.3}$$

Above $im_T(u, \mathbb{B}(x_0, r)) = \{y \in \mathbb{R}^2\backslash u(\partial\mathbb{B}(x_0, r)) :\ \deg(u, \mathbb{B}(x_0, r), y) \neq 0\}$.

6) If $|(\nabla u)^{-1}|^2 \det \nabla u \in L^1(\mathbf{X})$ *then* u *is a Sobolev homeomorphism with* $u^{-1} \in W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ *and* $\det \nabla u^{-1} \geq 0$ *for a.e.* $x \in \mathbf{X}$.

Above $(\nabla u)^{-1}$ *is the inverse of the matrix of weak derivatives of* u.

Proof. The proof of 1) follows from $\det \nabla u > 0$ *a.e.* in $\mathbf{X}$ with $u = x$ on $\partial\mathbf{X}$ and an application of Proposition 3.3 and Theorem 3.5 on pages 292-294 in [35]. To prove 2) we firstly use the fact that $\deg(u, \mathbf{X}, p) = \deg(x, \mathbf{X}, p)$ since $u(x) = x$ on $\partial\mathbf{X}$. Moreover as $\deg(x, \mathbf{X}, p) = 1$ for all $p \in \mathbf{X}$ and $\deg(x, \mathbf{X}, p) = 0$ for $p \in \mathbb{R}^2\backslash\overline{\mathbf{X}}$ we obtain $\deg(u, \mathbf{X}, p) = 1$ for all $x \in \mathbf{X}$ which implies that $\overline{\mathbf{X}} \subset u(\overline{\mathbf{X}})$. To see that $u(\overline{\mathbf{X}}) \subset \overline{\mathbf{X}}$ we can apply Theorem 5.35 in [34] to obtain that,

$$\deg(u, \mathbf{X}, p) = \int_{\mathbf{X}} f(u(x)) \det \nabla u \, dx, \tag{5.6.4}$$

for any $0 < f$, which is a continuous real valued function that satisfies $\int_{\mathbb{R}^2} f(x)dx = 1$ and has compact support in V the connect component of $\mathbb{R}^2\backslash\partial\mathbf{X}$ containing p. Now suppose that $\exists x \in \mathbf{X}$ such that $u(x) = p \in \mathbb{R}^2\backslash\overline{\mathbf{X}}$. Then pick $\delta > 0$ small enough such that $\overline{\mathbb{B}}_\delta(p) \subset \mathbb{R}^2\backslash\overline{\mathbf{X}}$ and let f have support on $\overline{\mathbb{B}}_\delta(p)$. The continuity of u means that $\exists\xi > 0$ such that $u(\mathbb{B}_\xi(x)) \subset \mathbb{B}_\delta(p)$ and therefore,

$$\deg(u, \mathbf{X}, p) = \int_{\mathbf{X}} f(u(x)) \det \nabla u \, dx > 0, \tag{5.6.5}$$

since $\det \nabla u > 0$ *a.e.* in $\mathbf{X}$. Hence we have reached a contradiction since $\deg(u, \mathbf{X}, p) = 0$ for $p \in \mathbb{R}^2\backslash\overline{\mathbf{X}}$ and thus $\overline{\mathbf{X}} = u(\overline{\mathbf{X}})$. To prove 3) we first note that by 2) we know that $N(u, \mathbf{X}, y) = \sharp\{x \in \mathbf{X} : u(x) = y\} \geq 1$ for $y \in \mathbf{X}$ and $N(u, \mathbf{X}, y) = 0$ for $y \in \mathbb{R}^2\backslash\overline{\mathbf{X}}$. Then by Theorem 5.34 in [34] we know that,

$$\int_{\mathbf{X}} f(u(x)) \det \nabla u \, dx = \int_{\mathbb{R}^2} N(u, \mathbf{X}, y) f(y) \, dy. \tag{5.6.6}$$

for any $f \in L^\infty(\mathbb{R}^2)$. Take $f = 1$ on $\mathbf{X}$ to obtain that,

$$|\mathbf{X}| = \int_{\mathbf{X}} \det \nabla u \, dx = \int_{\mathbf{X}} N(u, \mathbf{X}, y) \, dy. \tag{5.6.7}$$

Then as $N(u, \mathbf{X}, y) \geq 1$ we obtain the $N(u, \mathbf{X}, y) = 1$ for almost every $y \in \mathbf{X}$. In the case of 4) a proof can be found in [35], in particular pages 296-297 or [34] Theorem 5.32. The proof of 5) follows in two parts, namely the proof of a) and b). Firstly a) is a consequence of $\det \nabla u > 0$ *a.e.*, which leads to the monotonicity of u and this in an equivalent fashion shows that a) is satisfied. For more details on the reader is again referred to [35] and in particular pages 293-295. Lets us now focus on proving that b) holds. To this end fix a $x_0 \in \mathbf{X}$ and a $r \in (0, \operatorname{dist}(x_0, \partial\mathbf{X}))$. Now define $W \subset \mathbf{X}\backslash\mathbb{B}(x_0, r)$ such that

$$u(W) \subset im_T(u, \mathbb{B}(x_0, r)). \tag{5.6.8}$$

However we know by Lemma 3.2 in [35] page 291 that,

$$\text{for } a.e. \ y \in im_T(u, \mathbb{B}(x_0, r)) \ \exists x \in \mathbb{B}(x_0, r) \ \ s.t. \ \ u(x) = y. \tag{5.6.9}$$

Therefore (minus a set of measure zero) we have that $u(W) \subset \mathcal{S}$ where,

$$\mathcal{S} = \{y \in \mathbf{X} : u^{-1}(y) \text{ has more than one element } \}. \tag{5.6.10}$$

But from 3) we know that u is one-to-one almost everywhere so $|\mathcal{S}| = 0$ which implies that $|u(W)| = 0$. Therefore, (note $W \subset u^{-1}(u(W))$)

$$|W| = 0, \tag{5.6.11}$$

since u satisfies the N^{-1} property. Thus as $x_0 \in \mathbf{X}$ and $r \in (0, \operatorname{dist}(x_0, \partial\mathbf{X}))$ were arbitrary we conclude that *b*) holds which completes our proof of 5). Lastly to prove 6) we use a similar argument to that given in Remark 2.1 in [60] by the authors and for the ease of the reader we reduce this here. Take any $u \in \mathcal{A}^+(\mathbf{X})$ and begin by extending u by identity onto the ball $\mathbb{B}_{2R}$ where $R > b$, i.e. $\overline{\mathbf{X}} \subset \mathbb{B}_R$, and therefore $u \in \mathcal{A}^+(\mathbb{B}_{2R})$. Let us now note that in the case of 2×2 matrices A with $\det A > 0$, we have that $|A| = \det A|A^{-1}|$. Therefore,

$$\frac{|A|^2}{\det A} = \det A|A^{-1}|^2. \tag{5.6.12}$$

which implies that the assumption of $|(\nabla u)^{-1}|^2 \det \nabla u \in L^1(\mathbf{X})$ is equivalent to $K(x,u) = |\nabla u|^2/\det \nabla u \in L^1(\mathbf{X})$. This in turn gives that u is a mapping of L^1 integrable dilation and in a similar vain the extend u has dilation $K(x,u) \in L^1(\mathbb{B}_{2R})$. Then by Theorem 1 in [50] we know that there is a Stoilow's type factorisation of $u \in \mathcal{A}(\mathbb{B}_{2R})$. Namely, there exists a homeomorphism $h \in W^{1,2}(\Omega, \mathbb{B}_{2R})$ and a holomorphic mapping $\varphi \in (\Omega, \mathbb{C})$, where $\Omega \subset \mathbb{C}$ is some open set, such that,

$$u = \varphi \circ h^{-1}. \tag{5.6.13}$$

Furthermore, as $h^{-1}(\overline{\mathbb{B}}_R) \subset \Omega$ is a compact subset we know, since points where the derivative of φ vanishes are isolated, that φ restricted to $\overline{\mathbb{B}}_R$ is locally conformal at all but finitely many points. Thus, as h is a homeomorphism, u is a local homeomorphism at all but finitely many points on $\mathbb{B}_R$. Moreover (*see* [56]) since $\partial\mathbb{B}_R$ is connected, u is one-to-one on $\partial\mathbb{B}_R$ and a local homeomorphism at all but finitely many points we obtain that u is a global homeomorphism on $\mathbb{B}_R$, which in turn implies that u is a global homeomorphism on $\overline{\mathbf{X}}$. To obtain that $u^{-1} \in W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ we apply Theorem 6.1 in [39]. Note again by the assumption that $|(\nabla u)^{-1}|^2 \det \nabla u \in L^1(\mathbf{X})$ implies that the mapping u has dilation $K(x,u) \in L^1(\mathbf{X})$ so the assumptions of Theorem 6.1 are satisfied. Finally another consequence of Theorem 6.1 in [39] is that u^{-1} has finite dilation and thus $\det \nabla u^{-1} \geq 0$ for *a.e.* $x \in \mathbf{X}$. $\square$

With the relevant properties of the mappings $u \in \mathcal{A}^+(\mathbf{X})$ at hand let us describe the *annular rearrangement* which we alluded to earlier in this section.

Definition 5.6.1. (*Annular rearrangement*) *Let* $\mathbf{X} = \mathbf{X}[a,b]$ *be an annulus as before and* $0 \leq \alpha_1 \leq f \in \mathbf{C}(\overline{\mathbf{X}})$ *with* $f(x) = \alpha_1$ *when* $|x| = a$. *Then we define the annular rearrangement* f *to be given for* $x \in \mathbf{X}$ *to be,*

$$\mathbf{S}[f](x) = \alpha_2 - h^\sharp(x) \tag{5.6.14}$$

where $\alpha_2 = \sup_{\overline{\mathbf{X}}} f$ *and* $h^\sharp$ *denotes the Schwarz rearrangement of the function* h *which is defined by,*

$$h(x) = \begin{cases} \alpha_2 - f & \text{on } \mathbf{X}, \\ \alpha_2 - \alpha_1 & \text{on } \mathbb{B}_a. \end{cases} \tag{5.6.15}$$

Let us briefly recall for the ease of the reader that in this two dimensional case the Schwarz rearrangement of a function $h \in L^1(\mathbb{B}_b)$ is defined by,

$$h^\sharp(x) = \sup_t \{t \geq 0 : |\mathscr{A}_t[h]| > \pi |x|^2\}, \tag{5.6.16}$$

for $x \in \mathbb{B}_b$ and where $E_t[h] = \{x \in \mathbb{B}_b : h(x) > t\}$. We now state and prove a proposition which will be important in the main result of this section.

Proposition 5.6.3. *Let* $0 \leq \alpha_1 \leq f \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$ *with* $f = \alpha_1$ *when* $|x| = a$. *Then the annular rearrangement* $\mathbf{S}[f] \in W^{1,2}(\mathbf{X})$ *with* $\mathbf{S}[f](x) = 0$ *when* $|x| = a$ *and* $\mathbf{S}[f](x) = \alpha_2$ *when* $|x| = b$. *Additionally* $\mathbf{S}[f](x)$ *satisfies,*

$$\int_{\mathbf{X}} |\nabla f|^2 \, dx \geq \int_{\mathbf{X}} |\nabla \mathbf{S}[f](x)|^2 \, dx. \tag{5.6.17}$$

Proof. Firstly as $h \in W_0^{1,2}(\mathbb{B}_b)$ we know that $h^\sharp \in W_0^{1,2}(\mathbb{B}_b)$ and therefore $\mathbf{S}[f] \in W^{1,2}(\mathbf{X})$ with $\mathbf{S}[f](x) = \alpha_2$ when $|x| = b$. Further as,

$$\nabla h(x) = \begin{cases} -\nabla f & \text{on } \mathbf{X}, \\ 0 & \text{on } \mathbb{B}_a. \end{cases} \tag{5.6.18}$$

we obtain that

$$\int_{\mathbf{X}} |\nabla f|^2 \, dx = \int_{\mathbb{B}_b} |\nabla h|^2 \, dx \geq \int_{\mathbb{B}_b} |\nabla h^\sharp|^2 \, dx = \int_{\mathbf{X}} |\nabla \mathbf{S}[f]|^2 \, dx, \tag{5.6.19}$$

which gives (5.6.17). Finally since $|\{x \in \mathbb{B}_b : h = \alpha_2 - \alpha_1\}| \geq \pi a^2$ we obtain by the definition of the Schwarz rearrangement that $h^\sharp(x) = \alpha_2 - \alpha_1$ for $x \in \overline{\mathbb{B}}_b$ which in turn gives that $\mathbf{S}[f](x) = \alpha_1$ when $|x| = a$. $\square$

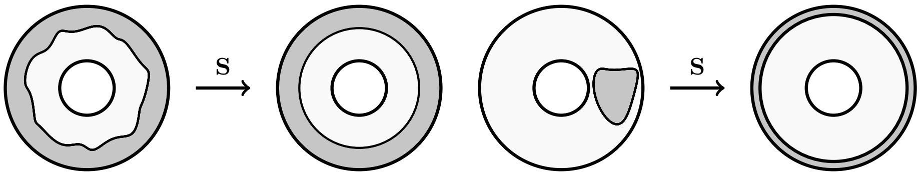


Figure 5.1: This figure shows how the *annular rearrangement* affects the level sets of a function $0 < \alpha_1 \leq f \in \mathbf{C}(\overline{\mathbf{X}})$ with $f = \alpha_1$ when $|x| = a$.

Proposition 5.6.4. *Given any mapping* $u \in \mathcal{A}^+(\mathbf{X})$ *the annular rearrangement of* $|u|$*, given by* $f(|x|) = \mathbf{S}[|u|](x)$ *belongs to* $\mathscr{F}_+$*, i.e.* $f \in \mathscr{F}_+$.

Proof. To begin we note that the previous proposition gives that $\mathbf{S}[|u|] \in W^{1,2}(\mathbf{X})$ with $\mathbf{S}[|u|](x) = a$ and $\mathbf{S}[|u|](x) = b$ for $|x| = a$ and $|x| = b$ respectively. Furthermore, by the definition of $\mathbf{S}[|u|]$ we see that $\mathbf{S}[|u|]$ is radially symmetric. Therefore, $\exists f \in W^{1,2}(a,b)$ such that $\mathbf{S}[|u|](x) = f(|x|)$ and further f is monotone increasing as the rearrangement $h^\sharp$ is monotone decreasing. Hence, in order to prove the above stated proposition we just need to show that $\dot{f}(r) > 0$ for *a.e.* in (a,b). In order to achieve this the idea is to show that,

$$|\{x \in \mathbf{X} : \nabla \mathbf{S}[|u|](x) = 0\}| = |\{x \in \mathbf{X} : \dot{f}(|x|) = 0\}| = 0. \tag{5.6.20}$$

To this end, note by the definition of $\mathbf{S}[|u|]$ that $\nabla \mathbf{S}[|u|] = -\nabla h^\sharp$ and therefore

$$\{x \in \mathbf{X} : \nabla \mathbf{S}[|u|](x) = 0\} = \{x \in \mathbf{X} : \nabla h^\sharp(x) = 0\}. \tag{5.6.21}$$

Additionally, $\nabla |u| = -\nabla h$ on $\mathbf{X}$ so

$$\{x \in \mathbf{X} : \nabla |u|(x) = 0\} = \{x \in \mathbf{X} : \nabla h(x) = 0\}, \tag{5.6.22}$$

and from the determinant constraint, namely $\det \nabla u > 0$ we can obtain that

$$0 < \det \nabla u = \partial_1 u \times \partial_2 u = \left\langle \left[u \times \partial_2 \left(\frac{u}{|u|} \right), u \times \partial_1 \left(\frac{u}{|u|} \right) \right]^t, \nabla |u| \right\rangle, \tag{5.6.23}$$

for *a.e.* $x \in \mathbf{X}$. The above very convenient form of the determinant is true in two dimensions and note that we have let ∂_j denote the partial derivative with respect to x_j, i.e. $\partial_j = \partial / \partial x_j$. Hence, $|\{x \in \mathbf{X} : \nabla |u| = 0\}| = 0$ and moreover,

$$|\{x \in \mathbf{X} : \nabla h(x) = 0\}| = 0. \tag{5.6.24}$$

Now since $h \in W_0^{1,2}(\mathbb{B}_b)$ we can use the coarea formula for Sobolev functions (*see*, e.g., [17]) to obtain that (below $\mu_h(t) = |\{x \in \mathbf{X} : h(x) > t\}|$),

$$\begin{aligned}\mu_h(t) &= |\{x \in \mathbf{X} : \nabla h = 0\} \cap h^{-1}(t, \infty)| + \int_t^b \int_{\{x \in \mathbf{X}: |u|=s\}} \frac{1}{|\nabla h|} \, d\mathcal{H}^1 \, ds \\ &= \int_t^b \int_{\{x \in \mathbf{X}: |u|=s\}} \frac{1}{|\nabla h|} \, d\mathcal{H}^1 \, ds. \end{aligned} \tag{5.6.25}$$

Thus, μ_h is absolutely continuous, which by arguing as in [17] Lemma 2.3 part (v) (or [22]) we can conclude that,

$$0 = |\{x \in \mathbf{X} : \nabla h^\sharp = 0\} \cap (h^\sharp)^{-1}(0, \infty)|. \tag{5.6.26}$$

This in turn gives that $|\{x \in \mathbf{X} : \nabla h^\sharp = 0\}| = 0$ and therefore by our earlier connection (5.6.21) we have that,

$$|\{x \in \mathbf{X} : \nabla \mathbf{S}[|u|] = 0\}| = 0. \tag{5.6.27}$$

Hence, as $\mathbf{S}[|u|](x) = f(|x|)$ and f is monotone increasing we have that $\dot{f} > 0$ for *a.e.* $a \leq r \leq b$. $\square$

Let us now describe how we can use the above annular rearrangement to define a corresponding twist mapping $\bar{u}$ from any mapping $u \in \mathscr{A}_k^+$ where $k \in \mathbb{Z}$. To this end let us first recall from Theorem 5.6.1 that each $u \in \mathscr{A}_k^+$ is of the form,

$$u(x) = |u| \mathrm{Q}[g]\theta, \tag{5.6.28}$$

where $|u|, g \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$ with $g(a\omega) = 0$ and $g(b\omega) = 2\pi k$ for all $\omega \in \mathbb{S}^1$. Now let the corresponding twist map be given by,

$$\bar{u}(x) = f(r) \mathrm{Q}[\bar{g}]\theta, \tag{5.6.29}$$

where as before we have let $f(r) = \mathbf{S}[|u|](x)$ and

$$\bar{g}(r) = \frac{1}{2\pi} \int_a^r \int_0^{2\pi} \frac{\partial g}{\partial r}(s, \vartheta) d\vartheta ds, \tag{5.6.30}$$

for $r \in (a, b)$. Moreover $\bar{g}$ defined by (5.6.30) satisfies $g \in W^{1,2}(a, b)$ and $g(a) = 0$ with $g(b) = 2\pi k$ for all $\omega \in \mathbb{S}^1$. Hence the twist mapping $\bar{u}$ defined by (5.6.29) lies in the same homotopy class, i.e. $\bar{u} \in \mathscr{A}_k^+$. In summary we have just proved the following,

Theorem 5.6.5. *Given any mapping $u \in \mathscr{A}_k^+$ for some $k \in \mathbb{Z}$ the twist mapping $\bar{u}$ defined* (5.6.29) *where $f(r) = \mathbf{S}[|u|]$ and $\bar{g}$ given by* (5.6.30) *satisfies $\bar{u} \in \mathscr{A}_k^+$.*

In the next section we establish some energy relations between u and the corresponding symmetrisation $\bar{u}$.

5.7 Twist mappings as L^1-local minimisers of the polyconvex energies $\mathbb{E}_\varepsilon[u]$

As the title of this section suggests we are going to prove that the twist mappings obtained from Theorem 5.5.4 in the section 5.5 are in fact L^1 -local minimisers to a wide variety of energies $\mathbb{E}_\varepsilon[u]$. To achieve this we show that the corresponding twist mappings given by $u_k^\varepsilon = f_k^\varepsilon \mathrm{Q}_k^\varepsilon \theta$ for each $k \in \mathbb{Z}$ and $\varepsilon > 0$ are minimisers of $\mathbb{E}_\varepsilon[u]$ in the homotopy classes $\mathscr{A}_k^+$. However before proceeding further we recall some general assumptions on the function F appearing in the energy $\mathbb{E}_\varepsilon[u]$ which we made in the latter part of Section 5.5. Namely that,

$$\mathrm{F}(x, y) = \varphi(x) y^{-1}, \tag{5.7.1}$$

where $0 < \varphi \in \mathbf{C}^\infty(0, \infty)$ is such that,

$$\frac{d^2}{dt^2}\left[\varphi^{1/2}(t^2)\right] \geq -\frac{d}{dt}\left[\frac{\varphi^{1/2}(t^2)}{t}\right] \tag{5.7.2}$$

for $0 < t$. An example of a φ satisfying the above constraint would be $\varphi(t) = t^\varsigma$ where $\varsigma^2 \geq 1$. Furthermore let us now consider the energy $\mathbb{E}_\varepsilon[u]$ as follows,

$$\mathbb{E}_\varepsilon[u] = \frac{1}{2}\int_{\mathbf{X}} \varphi(|x|^2)\frac{|\nabla u|^2}{|u|^2}\, dx + \frac{1}{\varepsilon}\int_{\mathbf{X}} h(\det \nabla u)\, dx =: \mathscr{D}[u] + \frac{1}{\varepsilon}\mathscr{H}[u]. \tag{5.7.3}$$

Here we have split the energy up into what we call its general distortion part $\mathscr{D}$ and its determinant part $\mathscr{H}$. In addition to our assumption on φ we shall also assume that the convex function h is such that the determinant energy $\mathscr{H}$ is continuous under L^1 convergence. Namely if we have that $\det \nabla u_n \to \det \nabla u$ as $n \to \infty$ in L^1 then $\lim_{n\to\infty} \mathscr{H}[u_n] = \mathscr{H}[u]$. One example where this is the case is if we simply assume that the convex function h satisfies $h(t) \leq c(1+t)$. With these assumptions at hand we are now in a position to prove the following result,

Proposition 5.7.1. *For any $k \in \mathbb{Z}$ and $u \in \mathscr{A}_k^+$ the corresponding twist mapping $\bar{u} \in \mathscr{A}_k^+$ defined by* (5.6.29) *satisfies,*

$$\mathscr{D}[\bar{u}] \leq \mathscr{D}[u]. \tag{5.7.4}$$

Namely the symmetrized mapping $\bar{u}$ has a general distortion energy no greater than that of u, whilst belonging to the same homotopy class.

Proof. To begin we recall that from Theorem 5.6.1 we know $\exists g \in \mathbf{C}(\mathbf{X})$ such that $g(b\omega) = 2\pi k$, $g(a\omega) = 0$ for all $\omega \in \mathbb{S}^1$ and u can be written in the following form $u(x) = |u|\mathrm{Q}[g]\theta$.

In particular,

$$|\nabla u|^2 = |\nabla |u||^2 + |x|^{-2}|u|^2\left(1 + 2\partial_\theta g\right) + |u|^2|\nabla g|^2, \tag{5.7.5}$$

for almost every $x \in \mathbf{X}$. Then using this identity we obtain that,

$$\begin{aligned} 2\mathscr{D}[u;\mathbf{X}] &= \int_{\mathbf{X}} \varphi(|x|^2)|u|^{-2}|\nabla u(x)|^2\, dx \\ &= \int_{\mathbf{X}} \varphi(|x|^2)\left[|u|^{-2}|\nabla |u||^2 + |x|^{-2}\left(1 + 2\partial_\theta g\right) + |\nabla g|^2\right]\, dx. \end{aligned} \tag{5.7.6}$$

Now let us also recall that $\bar{g}$ is defined be (5.6.30) in which g is written in terms of polar co-ordinates. As previously mentioned $\bar{g}(a) = 0$, $\bar{g}(b) = 2\pi k$ and $\bar{g} \in W^{1,2}(a,b)$. Therefore via an application of Jensen's inequality we obtain that,

$$|\nabla \bar{g}|^2 = |\frac{d\bar{g}}{dr}|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |\nabla g|^2 d\theta, \tag{5.7.7}$$

for almost every $r \in [a,b]$. Therefore,

$$\int_{\mathbf{X}} \varphi(|x|^2)|\nabla \bar{g}|^2\, dx \leq \int_{\mathbf{X}} \varphi(|x|^2)|\nabla g|^2\, dx. \tag{5.7.8}$$

It is also straightforward to see that,

$$\int_{\mathbf{X}} \varphi(|x|^2)|x|^{-2}\left(1 + 2\partial_\theta g\right)\, dx = \int_{\mathbf{X}} \varphi(|x|^2)|x|^{-2}\, dx. \tag{5.7.9}$$

Hence we have thus far shown that,

$$\int_{\mathbf{X}} \varphi(|x|^2)\left[|u|^{-2}|\nabla |u||^2 + |x|^{-2} + |\nabla \bar{g}|^2\right]\, dx \leq 2\mathscr{D}[u;\mathbf{X}], \tag{5.7.10}$$

and therefore we are left to show that the *annular rearrangement* $f = \mathbf{S}[|u|]$ satisfies,

$$\mathscr{K}[f;\mathbf{X}] := \int_{\mathbf{X}} \varphi(|x|^2)\frac{|\nabla |f||^2}{f^2}\, dx \leq \int_{\mathbf{X}} \varphi(|x|^2)\frac{|\nabla |u||^2}{|u|^2}\, dx = \mathscr{K}[|u|;\mathbf{X}]. \tag{5.7.11}$$

To this end we point out that if we extend u by identity onto the slightly larger annulus $\mathbf{X}_\delta = \{x \in \mathbb{R}^2 : a_\delta = a - \delta < |x| < b + \delta = b_\delta\}$ then $u \in \mathscr{A}_k^+(\mathbf{X}_\delta)$ and,

$$|u_\delta| = \begin{cases} |u| & x \in \mathbf{X}, \\ |x| & x \in \mathbf{X}_\delta \backslash \mathbf{X}. \end{cases} \implies f_\delta = \mathbf{S}[|u_\delta|] = \begin{cases} f & x \in \mathbf{X}, \\ |x| & x \in \mathbf{X}_\delta \backslash \mathbf{X}. \end{cases} \tag{5.7.12}$$

Therefore we trivially have the following equivalence,

$$\mathscr{K}[f_\delta;\mathbf{X}_\delta] \leq \mathscr{K}[|u_\delta|;\mathbf{X}_\delta] \iff \mathscr{K}[f;\mathbf{X}] \leq \mathscr{K}[|u|;\mathbf{X}]. \tag{5.7.13}$$

With this equivalence at hand we proceed to prove the left hand side of (5.7.13) for reasons which will become more apparent momentarily. Firstly we certainly know that the extended mapping u_δ satisfies $u_\delta(\overline{\mathbf{X}}_\delta)\cap u_\delta(\partial\mathbf{X}_\delta)=\emptyset$. Now by Theorem 1.1 in [41] we know that there exists a sequence of $\mathbf{C}^\infty$ diffeomorphisms $\{u_n\}$ such that $u_n = x$ on $\partial\mathbf{X}_\delta$ and $u_n \to u_\delta$ strongly in $W^{1,2}(\mathbf{X}_\delta,\mathbb{R}^2)$ and uniformly in $\mathbf{C}(\overline{\mathbf{X}}_\delta,\mathbb{R}^2)$. Consequently $|u_n|\to|u|$ strongly in $W^{1,2}(\mathbf{X}_\delta)$ and uniformly in $\mathbf{C}(\overline{\mathbf{X}}_\delta)$. Moreover by the identity boundary conditions of the diffeomorphisms u_n we know that $\det\nabla u_n > 0$ on $\mathbf{X}_\delta$ and therefore it is clear that $\{u_n\}\subset\mathcal{A}^+(\mathbf{X}_\delta)$. Hence we may firstly try to prove that (5.7.13) holds for the diffeomorphisms u_n.

To this end let f_n be that annular rearrangement of $|u_n|$, namely $f_n = \mathbf{S}[|u_n|]$. Then by Proposition 5.6.4 we know that $f_n\in\mathscr{F}_+$. Therefore we want to show that,

$$\int_{\mathbf{X}_\delta}\varphi(|x|^2)|f_n|^{-2}|\nabla f_n|^2dx \le \int_{\mathbf{X}_\delta}\varphi(|x|^2)|u_n|^{-2}|\nabla|u_n||^2\,dx. \tag{5.7.14}$$

First observe that since u_n is a diffeomorphism we know that $u_n(w_n(y)) = y$ for all $y\in\mathbf{X}_\delta$, where w_n is the inverse of u_n. Hence,

$$\{x\in\mathbf{X}_\delta : |u_n|>t\} = w_n(\mathbf{X}_t), \tag{5.7.15}$$

where $\mathbf{X}_t = \{x\in\mathbf{X}_\delta : |x|>t\}$ for $a<t<b$. Moreover we know that,

$$w_n(\overline{\mathbf{X}}_t) = \overline{w_n(\mathbf{X}_t)}, \quad w_n(\partial\mathbf{X}_t) = \partial w_n(\mathbf{X}_t). \tag{5.7.16}$$

Therefore using the identity boundary conditions,

$$\partial\{x\in\mathbf{X}_\delta : |u_n|>t\} = \{x\in\mathbf{X}_\delta : |x| = b_\delta\}\cup w_n(\{x\in\mathbf{X}_\delta : |x|=t\}), \tag{5.7.17}$$

and in particular,

$$w_n(\{x\in\mathbf{X}_\delta : |x|=t\}) = \{x\in\mathbf{X}_\delta : |u_n|=t\}. \tag{5.7.18}$$

Therefore as w_n is a smooth diffeomorphism we know that $\{x\in\mathbf{X}_\delta : |u_n|=t\}$ is diffeomorphic to a circle and hence for $t\in[a,b]$ we shall represent this smooth curve in polar co-ordinates by $\gamma_1(\theta) = w_n(r,\theta)$. Now let us define the function $\rho_n : [a_\delta,b_\delta]\to[a_\delta,b_\delta]$ to be,

$$\pi\rho_n(t)^2 = \pi b_\delta^2 - \int_{\mathbf{X}_t^{b_\delta}}\det\nabla w_n\,dx. \tag{5.7.19}$$

Clearly $\rho_n\in\mathbf{C}^\infty[a_\delta,b_\delta]$ and $\dot\rho_n>0$ as $\det\nabla w_n>0$. We also note that a straightforward change of variables gives that,

$$|\{x\in\mathbf{X}_\delta : |u_n(x)|>t\}| = \int_{\mathbf{X}_t^{b_\delta}}\det\nabla w_n\,dx = \pi\left[b_\delta^2-\rho_n(t)^2\right]. \tag{5.7.20}$$

Furthermore since the curve $\gamma_1(\theta)$ is smooth and diffeomorphic to the circle we obtain that,

$$\begin{aligned}\int_{\{x\in\mathbf{X}_\delta:|u_n|=t\}} \varphi(|x|^2)^{1/2}\,d\mathcal{H}^1 &= \int_0^{2\pi} \varphi(|x|^2)^{1/2}\left|\frac{\partial w_n}{\partial\theta}\right| d\theta \\ &\geq \int_0^{2\pi} \varphi(|x|^2)^{1/2}\frac{w_n\times\partial_\theta w_n}{|w_n|}\,d\theta, \end{aligned} \tag{5.7.21}$$

where the last inequality comes from the $|w_n||\partial_\theta w_n| \geq w_n\times\partial_\theta w_n$. Now as was proved by the authors in [58] Proposition 7.2 we have the following useful identity,

$$\int_0^{2\pi} \Gamma(|w_n|)\frac{(w_n\times\partial_\theta w_n)}{|w_n|^2}\,d\theta\bigg|_{a_\delta}^{t} = \int_{\mathbf{X}^t} |w_n|^{-1}\dot{\Gamma}(|w_n|)\det\nabla w_n\,dx, \tag{5.7.22}$$

where $\Gamma(t)\in\mathbf{C}^1[a_\delta,b_\delta]$ and where here $\mathbf{X}^t=\{x\in\mathbf{X}_\delta : |x|<t\}$. Note that Proposition 7.2 [58] is actually for mappings $u\in\mathcal{A}(\mathbf{X})$ but the proof works by approximation by diffeomorphisms as here and in particular (5.7.22) is shown directly in the proof of Proposition 7.2 by equation (7.9) on page 19. Note that in the notation of [58] we have that $\Gamma(t)=t^2\Phi(t)^2$. Here we let $\Gamma(t)=\varphi(t^2)^{1/2}t$ then (5.7.21) becomes,

$$2\pi\Gamma(a_\delta)+\int_{\mathbf{X}^t}|w_n|^{-1}\dot{\Gamma}(|w_n|)\det\nabla w_n\,dx \leq \int_{\{x\in\mathbf{X}_\delta:|u_n|=t\}}\varphi(|x|^2)^{1/2}\,d\mathcal{H}^1,$$

Then again using a change of variables we find that,

$$2\pi\Gamma(a_\delta)+\int_{w_n(\mathbf{X}^t)}|y|^{-1}\dot{\Gamma}(|y|)\,dy \leq \int_{\{x\in\mathbf{X}_\delta:|u_n|=t\}}\varphi(|x|^2)^{1/2}\,d\mathcal{H}^1. \tag{5.7.23}$$

Let us now recall that $w_n(\mathbf{X}_t)=\{x\in\mathbf{X}_\delta : |u_n|>t\}=E_t$ and,

$$\mathbf{X}_\delta = w_n(\mathbf{X}_t)\cup w_n(\mathbf{X}^t)\cup w_n(\{x\in\mathbf{X} : |x|=t\}), \tag{5.7.24}$$

since by the Jordan-Schöenflies theorem $w_n(\{x\in\mathbf{X} : |x|=t\})$ partitions $\mathbb{R}^2$ into two components. This therefore gives that,

$$\begin{aligned}\int_{w_n(\mathbf{X}^t)}|y|^{-1}\dot{\Gamma}(|y|)\,dy &= \int_{\mathbf{X}_\delta}|y|^{-1}\dot{\Gamma}(|y|)\,dy - \int_{w_n(\mathbf{X}_t)}|y|^{-1}\dot{\Gamma}(|y|)\,dy \\ &= 2\pi\left[\Gamma(b_\delta)-\Gamma(a_\delta)\right] - \int_{w_n(\mathbf{X}_t)}|y|^{-1}\dot{\Gamma}(|y|)\,dy. \end{aligned} \tag{5.7.25}$$

At this point we note that the assumptions we have placed on φ ensure that $\Gamma\in\mathbf{C}^2[a_\delta,b_\delta]$ and that $|y|^{-1}\dot{\Gamma}(|y|)$ is monotone increasing. Hence,

$$\begin{aligned}\int_{w_n(\mathbf{X}_t)}|y|^{-1}\dot{\Gamma}(|y|)\,dy &= \int_{\mathbf{X}_\delta}\chi_{E_t}|y|^{-1}\dot{\Gamma}(|y|)\,dy \\ &\leq \int_{\mathbf{X}_\delta}\chi_{E_t^*}|y|^{-1}\dot{\Gamma}(|y|)\,dy. \end{aligned} \tag{5.7.26}$$

Now we also recall that $|E_t| = \pi\left[b_\delta^2 - \rho_n(t)^2\right]$ which gives that,

$$\int_{\mathbf{X}_\delta} \chi_{E_t^*}|y|^{-1}\dot{\Gamma}(|y|)\,dy = \int_0^{2\pi}\int_{\rho_n(t)}^{b_\delta} \dot{\Gamma}(|y|)\,dr\,d\theta = 2\pi[\Gamma(b_\delta) - \Gamma(\rho_n(t))]. \tag{5.7.27}$$

Hence combining the above with (5.7.26) allows us to conclude that,

$$\int_{w_n(\mathbf{X}^t)} |y|^{-1}\dot{\Gamma}(|y|)\,dy \geq 2\pi[\Gamma(\rho_n(t)) - \Gamma(a_\delta)]. \tag{5.7.28}$$

Then substituting this into (5.7.23) we have that,

$$2\pi\varphi(\rho_n(t))^{1/2}\rho_n(t) = 2\pi\Gamma[\rho_n(t)] \leq \int_{\{x\in\mathbf{X}_\delta:|u_n|=t\}} \varphi(|x|^2)^{1/2}\,d\mathcal{H}^1. \tag{5.7.29}$$

Moreover an application of the coarea formula gives that

$$\int_{\{x\in\mathbf{X}_\delta:|u_n|=t\}} \frac{1}{|\nabla|u_n||}\mathrm{d}\mathcal{H}^1 = -\frac{d}{dt}\alpha_{|u_n|}(t) = 2\pi\dot{\rho}_n(t)\rho_n(t), \tag{5.7.30}$$

for *a.e.* $t \in (a_\delta, b_\delta)$, where again we recall that $\alpha_{|u_n|}(t) = |\{x \in \mathbf{X}_\delta : |u_n(x)| > t\}| = \pi(b_\delta^2 - \rho_n^2)$. Then by Holder's inequality and an application of (5.7.29) we get that,

$$2\pi\frac{\varphi(\rho_n^2)\rho_n(t)}{t^2\dot{\rho}_n(t)} \leq \int_{\{x\in\mathbf{X}_\delta:|u_n|=t\}} t^{-2}\varphi(|x|^2)|\nabla|u_n||\,\mathrm{d}\mathcal{H}^1, \tag{5.7.31}$$

for *a.e.* $t \in (a_\delta, b_\delta)$. However, recall that $f_n = \mathbf{S}[|u_n|]$ is defined such that $f_n(r) = \sup\{t : r > \rho_n(t)\}$ which clearly gives that $f_n(\rho_n(t)) = t$ and then an application of the change rule gives that,

$$2\pi\frac{\varphi(\rho_n^2)\rho_n\dot{f}_n(\rho_n)}{t^2} \leq \int_{\{x\in\mathbf{X}_\delta:|u_n|=t\}} t^{-2}\varphi(|x|^2)|\nabla|u_n||\,\mathrm{d}\mathcal{H}^1, \tag{5.7.32}$$

for almost every $t \in (a_\delta, b_\delta)$. Hence applying the coarea formula to $|u_n|$ we obtain that,

$$\begin{aligned}
\int_{\mathbf{X}_\delta} \varphi(|x|^2)f_n^{-2}|\nabla f_n|^2\,dx &= \int_a^b\int_{\{f_n=t\}} \varphi(|x|^2)f_n^{-2}|\nabla f_n|d\mathcal{H}^1dt \\
&= 2\pi\int_a^b t^{-2}\rho_n\varphi(\rho_n^2)\dot{f}_n(\rho_n)dt \\
&\leq \int_{\mathbf{X}_\delta} \varphi(|x|^2)|u_n|^{-2}|\nabla|u_n|^2|\,dx.
\end{aligned} \tag{5.7.33}$$

Its not too difficult to see from (5.7.33) that $\{f_n\}$ is bounded in $W^{1,2}(a_\delta, b_\delta)$ and therefore on a subsequence $f_n \rightharpoonup f^*$ in $W^{1,2}(a_\delta, b_\delta)$. However we already know that the annular rearrangement is non-expansive in L^2 so $f_n \to f_\delta$ in L^2 and therefore $f^* = f_\delta$. Furthermore by sequentially weak lower semi-continuity of $\mathscr{K}[\cdot;\mathbf{X}_\delta]$ we know that,

$$\begin{aligned}
\int_{\mathbf{X}_\delta} \varphi(|x|^2)f_\delta^{-2}|\nabla f_\delta|^2\,dx &\leq \lim_{n\to\infty}\inf\int_{\mathbf{X}_\delta} \varphi(|x|^2)f_n^{-2}|\nabla f_n|^2\,dx \\
&\leq \lim_{n\to\infty}\inf\int_{\mathbf{X}_\delta} \varphi(|x|^2)|u_n|^{-2}|\nabla|u_n|^2|\,dx \\
&= \int_{\mathbf{X}_\delta} \varphi(|x|^2)|u_\delta|^{-2}|\nabla|u_\delta|^2|\,dx.
\end{aligned} \tag{5.7.34}$$

Note that the last equality comes from $|u_n| \to |u_\delta|$ strongly in $W^{1,2}(\mathbf{X}_\delta)$ and uniformly in $\mathbf{C}(\overline{\mathbf{X}}_\delta)$. Hence as previous noted (5.7.34) implies that $\mathscr{K}[f;\mathbf{X}] \leq \mathscr{K}[|u|;\mathbf{X}]$ which combined with (5.7.10) gives that,

$$\mathscr{D}[\bar{u};\mathbf{X}] \leq \mathscr{D}[u;\mathbf{X}], \tag{5.7.35}$$

and therefore we conclude our proof. □

With this previous proposition in mind we now want to know if the corresponding twist map $\bar{u}$ also decreases the determinant energy. Namely is it also true the $\mathscr{H}[\bar{u}] \leq \mathscr{H}[u]$. The next proposition show us that this in fact the case. The proof of the result is inspired by the symmetrisation argument used in [69].

Proposition 5.7.2. *Consider the determinant energy given by $\mathscr{H}$ and let $u \in \mathscr{A}_k^+$. Then the twist mapping $\bar{u}$ satisfies,*

$$\mathscr{H}[\bar{u};\mathbf{X}] \leq \mathscr{H}[u;\mathbf{X}]. \tag{5.7.36}$$

Proof. Firstly let us, as in the proof of the previous proposition, extend our mapping u by identity onto $\mathbf{X}_\delta$ and approximate using a sequence of smooth diffeomorphisms $\{u_n\}$ as in Theorem 1.1 of [41]. Moreover, like the proof of the previous proposition the following are equivalent,

$$\mathscr{H}[\bar{u}_\delta;\mathbf{X}_\delta] \leq \mathscr{H}[u_\delta;\mathbf{X}_\delta] \iff \mathscr{H}[\bar{u};\mathbf{X}] \leq \mathscr{H}[u;\mathbf{X}]. \tag{5.7.37}$$

Furthermore we let w_n again denote the inverse of the diffeomorphism u_n. Then a change in variable formula gives that,

$$\begin{aligned} \int_{\mathbf{X}_\delta} h(\det \nabla u_n)\, dx &= \int_{\mathbf{X}_\delta} h(\det \nabla u_n(w_n)) \det \nabla w_n \, dy \\ &= \int_{\mathbf{X}_\delta} h((\det \nabla w_n)^{-1}) \det \nabla w_n \, dy. \end{aligned} \tag{5.7.38}$$

Now define $\bar{w}_n$ to be $\bar{w}_n(x) = \rho_n(r)\mathrm{Q}[\tilde{g}]\theta$ where $\tilde{g}(r) = -\bar{g}(\rho_n(r))$ and ρ_n is by

$$\pi\rho_n(r)^2 = \pi b^2 - \int_{\mathbf{X}_r^{b_\delta}} \det \nabla w_n \, dx. \tag{5.7.39}$$

A direct calculation shows that for $r \in (a_\delta, b_\delta)$,

$$\det \nabla \bar{w}_n = \frac{1}{2\pi} \int_0^{2\pi} \det \nabla w_n \, d\theta. \tag{5.7.40}$$

Now due to the fact the function $\psi(t) = h(t^{-1})t$ is convex, as $h(\cdot)$ is convex, we obtain via an application of Jensen's inequality that,

$$\int_{\mathbf{X}_\delta} h((\det \nabla w_n)^{-1}) \det \nabla w_n \, dy \geq \int_{\mathbf{X}_\delta} h((\det \nabla \bar{w}_n)^{-1}) \det \nabla \bar{w}_n \, dy. \tag{5.7.41}$$

In order to complete the proof we must show that,

$$\int_{\mathbf{X}_\delta} h((\det \nabla \bar{w}_n)^{-1}) \det \nabla \bar{w}_n \, dy = \int_{\mathbf{X}_\delta} h(\det \nabla \bar{u}_n) \, dy. \tag{5.7.42}$$

To achieve this we firstly recall that $\det \nabla \bar{w}_n = \rho_n \dot{\rho}_n / r$ for *a.e.* $r \in (a_\delta, b_\delta)$. Therefore,

$$\int_{\mathbf{X}_\delta} h((\det \nabla \bar{w}_n)^{-1}) \det \nabla \bar{w}_n \, dy = \int_{\mathbf{X}_\delta} h(|y|(\rho_n \dot{\rho}_n)^{-1}) \rho_n \dot{\rho}_n |y|^{-1} \, dy. \tag{5.7.43}$$

Furthermore since $\bar{u}_n \in \mathcal{A}^+(\mathbf{X}_\delta)$ we know, *see* Theorem 5.34 on page 145 in [34], for any $v \in L^\infty(\mathbf{X}_\delta)$ that,

$$\int_{\mathbf{X}_\delta} v(\bar{u}_n) \det \nabla \bar{u}_n \, dx = \int_{\mathbf{X}_\delta} v(y) \, dy. \tag{5.7.44}$$

Additionally we note that $\det \nabla \bar{u}_n = f_n \dot{f}_n / r$ and $\rho_n \in \mathbf{C}^\infty[a_\delta, b_\delta]$ which implies

$$h((\det \nabla \bar{w}_n)^{-1}) \det \nabla \bar{w}_n = h(|y|(\rho_n \dot{\rho}_n)^{-1}) \rho_n \dot{\rho}_n / r \tag{5.7.45}$$

is uniformly continuous on $\overline{\mathbf{X}}_\delta$. Therefore an application of the above change in variable formula gives,

$$\int_{\mathbf{X}_\delta} h((\det \nabla \bar{w}_n)^{-1}) \det \nabla \bar{w}_n \, dy = \int_{\mathbf{X}_\delta} h\left(\frac{f_n}{\rho_n(f_n)\dot{\rho}_n(f_n)}\right) \frac{\rho_n(f_n)\dot{\rho}_n(f_n)\dot{f}_n}{|x|} \, dx. \tag{5.7.46}$$

Moreover since u_n is a diffeomorphism we know that,

$$\begin{aligned} \pi b_\delta^2 - \pi \rho_n(f_n(r))^2 &= |\{x \in \mathbf{X}_\delta : |u_n| > f_n(r)\}| \\ &= |\{x \in \mathbf{X}_\delta : |\bar{u}_n| > f_n(r)\}| \\ &= |\{x \in \mathbf{X}_\delta : f_n(|x|) > f_n(r)\}| \\ &= |\{x \in \mathbf{X}_\delta : |x| > r\}| = \pi b_\delta^2 - \pi r^2, \end{aligned} \tag{5.7.47}$$

where the penultimate equality comes from f_n being strictly increasing on $[a_\delta, b_\delta]$. Therefore $\rho_n(f_n(r)) = r$ for $r \in (a_\delta, b_\delta)$ and by the chain rule $\dot{\rho}_n(f_n(r))\dot{f}_n(r) = 1$ for *a.e.* $r \in (a_\delta, b_\delta)$. Hence (5.7.46) becomes,

$$\begin{aligned} \int_{\mathbf{X}_\delta} h((\det \nabla \bar{w}_n)^{-1}) \det \nabla \bar{w}_n \, dy &= \int_{\mathbf{X}_\delta} h\left(\frac{f_n \dot{f}_n}{|x|}\right) dx \\ &= \int_{\mathbf{X}_\delta} h\left(\det \nabla \bar{u}_n\right) dx \end{aligned} \tag{5.7.48}$$

Hence combining this with (5.7.41) gives that,

$$\mathscr{H}[\bar{u}_n; \mathbf{X}_\delta] = \int_{\mathbf{X}_\delta} h(\det \nabla \bar{u}_n) \, dx \leq \int_{\mathbf{X}_\delta} h(\det \nabla u_n) \, dx = \mathscr{H}[u_n; \mathbf{X}_\delta]. \tag{5.7.49}$$

To finish we must note again (as in the proof of Proposition 5.7.1) that $f_n \to f_\delta$ in $L^2(a_\delta, b_\delta)$ and $f_n \rightharpoonup f_\delta$ in $W^{1,2}(a_\delta, b_\delta)$ from which we obtain that $\dot{f}_n f_n/r \rightharpoonup \dot{f}_\delta f_\delta/r$ in $L^1(a_\delta, b_\delta)$. Therefore,

$$\mathscr{H}[\bar{u}_\delta; \mathbf{X}_\delta] \leq \lim_{n\to\infty} \mathscr{H}[\bar{u}_n; \mathbf{X}_\delta] \leq \lim_{n\to\infty} \mathscr{H}[u_n; \mathbf{X}_\delta]. \tag{5.7.50}$$

Finally since $u_n \to u_\delta$ as $n \to \infty$ strongly in $W^{1,2}(\mathbf{X}_\delta, \mathbb{R}^2)$ we obtain that $\det \nabla u_n \to \det \nabla u_\delta$ in $L^1(\mathbf{X}_\delta)$. Thus, $\lim_{n\to\infty} \mathscr{H}[u_n; \mathbf{X}_\delta] = \mathscr{H}[u; \mathbf{X}_\delta]$ by our assumption on $\mathscr{H}$ and therefore,

$$\mathscr{H}[\bar{u}_\delta; \mathbf{X}_\delta] \leq \mathscr{H}[u_\delta; \mathbf{X}_\delta], \tag{5.7.51}$$

which by (5.7.37) completes our proof. $\square$

Now we have these two previous propositions at hand the fact that $(f^k_\varepsilon, \mathrm{Q}^k_\varepsilon)$, from Theorem 5.5.4, gives rise to a corresponding twist $u^k_\varepsilon = f^k_\varepsilon \mathrm{Q}^k_\varepsilon \theta$ that is a minimiser in $\mathcal{A}^+_k$ is a straightforward consequence.

Theorem 5.7.3. *For each $\varepsilon > 0$ and $k \in \mathbb{Z}$ the pair $(f^k_\varepsilon, \mathrm{Q}^k_\varepsilon) \in \mathscr{W}_k$ such that,*

$$\mathbb{K}_\varepsilon[f^k_\varepsilon, \mathrm{Q}^k_\varepsilon] = \inf_{\mathscr{W}_k} \mathbb{K}_\varepsilon, \tag{5.7.52}$$

define a twist $u^k_\varepsilon = f^k_\varepsilon \mathrm{Q}^k_\varepsilon \theta \in \mathscr{A}^+_k$ such that,

$$\mathbb{E}_\varepsilon[u^k_\varepsilon] = \inf_{u \in \mathscr{A}^+_k} \mathbb{E}_\varepsilon[u]. \tag{5.7.53}$$

Moreover u^k_ε is a L^1-local minimiser in $\mathcal{A}^+(\mathbf{X})$.

Proof. Firstly from Proposition 5.7.1 and Proposition 5.7.2 we know that for any $u \in \mathscr{A}^+_k$ the twist $\bar{u} \in \mathscr{A}^+_k$ defined by (5.6.29) satisfies,

$$\mathbb{E}_\varepsilon[\bar{u}] \leq \mathbb{E}_\varepsilon[u]. \tag{5.7.54}$$

Furthermore, by (5.2.4) we know that $\mathbb{E}_\varepsilon[\bar{u}] = \eta \mathbb{K}_\varepsilon[f, \mathrm{Q}]$, where f and Q are defined as in (5.6.29). Therefore,

$$\mathbb{E}[u^k_\varepsilon] = \eta \mathbb{K}_\varepsilon[f^k_\varepsilon, \mathrm{Q}^k_\varepsilon] = \eta \inf_{(f,\mathrm{Q}) \in \mathscr{W}_k} \mathbb{K}_\varepsilon[f, \mathrm{Q}] \leq \inf_{u \in \mathscr{A}^+_k} \mathbb{E}_\varepsilon[u], \tag{5.7.55}$$

which gives that u^k_ε is a minimiser in the homotopy class $\mathscr{A}^+_k$ and consequently a L^1-local minimiser in $\mathcal{A}^+(\mathbf{X})$ (*see* [74] or [76]). $\square$

Chapter 6

On Weyl's asymptotics and remainder term for the orthogonal and unitary groups

Abstract

We examine the asymptotics of the spectral counting function of a compact Riemannian manifold by V.G. Avakumovic [4] and L. Hörmander [43] and show that for the scale of orthogonal and unitary groups $\mathbf{SO}(N)$, $\mathbf{SU}(N)$, $\mathbf{U}(N)$ and $\mathbf{Spin}(N)$ it is not sharp. While for negative sectional curvature improvements are possible and known, *cf.* e.g., J.J. Duistermaat & V. Guillemin [28], here, we give sharp and contrasting examples in the positive Ricci curvature case [non-negative for $\mathbf{U}(N)$]. Furthermore here the improvements are sharp and quantitative relating to the dimension and *rank* of the group. We discuss the implications of these results on the closely related problem of closed geodesics and the length spectrum.

Contents

6.1 Introduction

Let (M^d, g) be a d dimensional compact Riemannian manifold without boundary and let $-\Delta_g$ denote the Laplace-Beltrami operator on M^d. The spectral counting function of $-\Delta_g$ on M^d is the function defined on $(0, \infty)$ by

$$\mathscr{N}(\lambda; M^d) = \sharp\{j \geq 0 : \lambda_j \leq \lambda\}, \qquad \lambda > 0. \tag{6.1.1}$$

In the sequel when the choice of M^d is clear from the context, or when there is no danger of confusion, we suppress the dependence on M^d and simply write $\mathscr{N} = \mathscr{N}(\lambda)$. Here $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$ denote the eigen-values of $-\Delta_g$ in ascending order where by basic spectral theory each eigen-value has a finite multiplicity while $\lambda_j \nearrow \infty$ as $j \nearrow \infty$. The description of the asymptotics of the spectral counting function has been the subject of numerous investigations and is a result of a sequence of improvements and refinements starting originally from the seminal works of H. Weyl in 1911 describing the leading term and then gradually sharpening the form and order of the remainder term through the works of various authors most notably B.M. Levitan [52], V.G. Avakumovic [4] and L. Hörmander [43]. (See also the monographs V. Ivrii [47] and M.A. Shubin [67].) For compact boundaryless manifolds the celebrated Avakumovic-Hörmander-Weyl asymptotics has the from

$$\mathscr{N}(\lambda; M^d) = \frac{\mathrm{Vol}_g(M^d)\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{d-1}{2}}\right), \qquad \lambda \nearrow \infty, \tag{6.1.2}$$

where $\mathrm{Vol}_g(M^d)$ is the volume of M^d with respect to dv_g and ω_d is the volume of the unit d-ball in the Euclidean space $\mathbb{R}^d$, that is, $\omega_d = |\mathbb{B}_1^d|$. The formulation of (6.1.2) is originally due to V.G. Avakumovic [4] and later on L. Hörmander [43], who by invoking the theory of Fourier integral operators and the wave equation gave a proof that simultaneously extends to operators of arbitrary order (*see* [27, 44], [45] Vols. **3-4** or [67]). That the remainder term is sharp can be seen by examining, e.g., Euclidean spheres or projective spaces (see below for more on this) whilst in contrast, for flat tori, the problem directly relates to counting integer lattice points and is far from sharp. Indeed recall that for $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the eigen-functions $(\varphi_\alpha : \alpha \in \mathbb{Z}^d)$ are characterised by

$$\begin{aligned} -\Delta_g \varphi_\alpha = 4\pi^2|\alpha|^2\varphi_\alpha, &\qquad \varphi_\alpha = e^{2\pi i \alpha\cdot x}, \\ \alpha = (\alpha_1, ..., \alpha_d), &\qquad |\alpha|^2 = \alpha_1^2 + ... + \alpha_d^2, \end{aligned} \tag{6.1.3}$$

and so as a result for $\lambda > 0$ and with χ the characteristic function of the closed ball centred at the origin and radius $r = \sqrt{\lambda}/2\pi$ we have

$$\begin{aligned} \mathscr{N}(\lambda; \mathbb{T}^d) &= \sharp\left\{x \in \mathbb{Z}^d : 4\pi^2|x|^2 \le \lambda\right\} = \sum_{x\in\mathbb{Z}^d} \chi(x) \\ &= \frac{\mathrm{Vol}_g(\mathbb{T}^d)\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{d-1}{2} - \frac{1}{2}\frac{d-1}{d+1}}\right). \end{aligned} \tag{6.1.4}$$

For further improvements of this classical result of E. Hlawka [42] and more on the lattice point problem see F. Chamizo & H. Iwaniec [20], D.R. Heath-Brown [38], M.N. Huxley [46], A. Walfisz [80] as well as F. Fricker [32]. See also Section 6.2 and Table 6.1 below.

A natural question arising from this is when is the remainder term in (6.1.2) sharp and if the sharpness of this term carries any geometric information. To elaborate on this further consider the half-wave equation

$$\begin{cases} \partial_t u + iAu = 0 & t \in \mathbb{R}, \\ u = f & t = 0, \end{cases} \tag{6.1.5}$$

associated with $A = \sqrt{-\Delta_g}$ in $L^2(M)$. Then $u(t) = e^{-itA}f$ and so upon taking traces and with $\mathscr{N}_A(\mu) = \sharp\{j : \mu_j = \sqrt{\lambda_j} \le \mu\} = \mathscr{N}(\mu^2)$ we have

$$\mathrm{Tr}\left(e^{-it\sqrt{-\Delta_g}}\right) = \sum e^{-it\sqrt{\lambda_j}} = \widehat{d\mathscr{N}_A}(t/2\pi). \tag{6.1.6}$$

The half-wave propagator $e^{-it\sqrt{-\Delta_g}}$ can be expressed as a Fourier integral operator and in view of the above identity its trace is the Fourier transform of the measure $d\mathscr{N}_A$.

Thus by basic considerations the asymptotics of this trace near $t = 0$ translates via a Fourier inversion (a trivial but more revealing Tauberian theorem in this context) to the asymptotics of the spectral counting function as $\lambda \nearrow \infty$.

Now consideration of this Fourier integral operator, its canonical relation and the Lagrangian flow associated to its principal symbol leads to the geodesic flow on the cotangent bundle $T^\star M$ of M (note that it is precisely here that one needs to deal with the first order operator $\sqrt{-\Delta_g}$). The periodic orbits of this Lagrangian flow are the periodic geodesics on M and the resulting length spectrum contains the singular support of the distributional trace

$$\rho(t) = \mathrm{Tr}(e^{-it\sqrt{-\Delta_g}}) = \sum_j e^{-it\sqrt{\lambda_j}}, \qquad -\infty < t < \infty. \tag{6.1.7}$$

Indeed the analysis by L. Hörmander of the big singularity of ρ at $t = 0$ leads to the trace formula of J.J. Duistermaat & V. Guillemin [28]

$$\sum_j h(\mu - \mu_j) \cong \frac{1}{(2\pi)^d} \sum_{k=0}^{d-1} c_k \mu^{d-1-k}, \qquad c_k = \int_{M^d} \omega_k, \qquad \mu \nearrow \infty, \tag{6.1.8}$$

where $\mu_j = \sqrt{\lambda_j}$ and $h \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp} \hat{h} \subset [-\varepsilon, \varepsilon]$ while $h \equiv 1$ in a suitably small neighbourhood of zero and ω_k smooth real-valued densities on M^d associated to the metric g. (In particular we have $c_0 = \mathrm{Vol}(\mathbb{B}^\star M)$ the volume of the unit ball in the co-tangent bundle.)

The sharpness of the remainder term in (6.1.2) now connects directly with the structure of the spectrum $(\lambda_j : j \geq 0)$ and the nature of the geodesic flow. For example in case of Euclidean spheres, real or complex projective spaces or more generally compact rank one symmetric spaces the spectrum clusters, the geodesic flow is periodic and the remainder term in (6.1.2) is sharp. However, and in contrast, for spaces with non-positive sectional curvature or more generally spaces with measure-theoretically *few* periodic geodesics the remainder term in (6.1.2) is *not* sharp and can be improved to $O(\lambda^{(d-1)/2}/\log \lambda)$ and $o(\lambda^{(d-1)/2})$ respectively. (*See* P. Berard [14], J.J. Duistermaat & V. Guillemin [28] as well as [5], V. Ivrii [47], C. Sogge [71] for more.)

In this paper motivated, by the significance of the period geodesics on the Lie group $\mathbb{G} = \mathbf{SO}(N)$ in providing twist solutions to certain geometric problems in the calculus of variations (*see* [64, 75, 77]) we take a closer look at the geodesic length spectrum and the spectral counting function and examining the sharpness of the remainder term in Weyl's law (6.1.2) for this case as well as the cases of the unitary and spinor groups. By analogy with the case of symmetric spaces and in contrast to the negative curvature case above, one expects, in virtue of positivity of $\mathrm{Ric}(\mathbb{G})$, that again the remainder term is sharp, however, we show this *not* to be the case. As a prototype example for the special orthogonal group $\mathbf{SO}(N)$ apart from $N = 3$ where the geodesic flow is periodic – note that $\mathbf{SO}(3) \cong \mathbb{P}(\mathbb{R}^3)$ is a rank one symmetric space – the remainder term in (6.1.2) is not sharp and can be quantitatively improved.

Indeed the spectral counting function $\mathscr{N} = \mathscr{N}(\lambda; \mathbb{G})$ of the orthogonal and unitary group $\mathbb{G}$ equipped with a bi-invariant metric g has the asymptotics:

$$\mathscr{N}(\lambda; \mathbb{G}) = \frac{\omega_d \mathrm{Vol}_g(\mathbb{G})}{(2\pi)^d} \lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{1}{2}[d-1-\varepsilon]}\right), \qquad \lambda \nearrow \infty, \tag{6.1.9}$$

where $d = \dim(\mathbb{G})$, $n = \mathrm{rank}(\mathbb{G})$ (*see* Tables 6.2 & 6.3 below) and $\varepsilon = \varepsilon(n) \geq 0$. In fact we show that $\varepsilon = 1$ when $n \geq 8$ and more generally $\varepsilon = (n-1)/(n+1)$ when $n \geq 1$. In particular when $\mathrm{rank}(\mathbb{G}) \geq 2$ the Avakumovic-Hörmander-Weyl remainder term in (6.1.2) is not sharp and as direct calculation (as in Section 6.2 below) reveals this is precisely when the geodesic flow of $\mathbb{G}$ *fails* to be periodic. More interestingly when $\mathrm{rank}(\mathbb{G}) \geq 8$ the exponent of λ in the remainder term (6.1.2) can be improved to $(d-2)/2$ which is sharp.

Let us end this introduction by giving a brief plan of the paper. In Section 6.2 we go over the main results and tools from the representation theory of compact Lie groups, in particular the computation of the Casimir spectrum, root systems and the analytic weights of irreducible representations followed by calculations relating to periodic geodesics and the length spectrum. In the interest of brevity and for the sake of definiteness the discussion here is confined to the prototype case of the special orthogonal group $\mathbf{SO}(N)$. In Sections 6.3 and 6.4 we give detailed analysis of the spectral counting function and its asymptotics for the scale of special orthogonal, unitary and spinor groups. Finally in Section 6.5 we present and prove the sharper form of the asymptotics of $\mathscr{N} = \mathscr{N}(\lambda; \mathbb{G})$ as highlighted in the discussion above for $n \geq 5$.

6.2 Weyl chambers and spectral multiplicities for the Lie group $\mathbb{G} = \mathbf{SO}(N)$ with $N \geq 2$

In this section we gather together some of the technical apparatus for computing and describing the spectrum and spectral multiplicities of the Laplace-Beltrami operator required later for the development of the paper. The reader is referred to the monographs [16, 40] and [51] for further details and the jargon on Lie groups and their representations. Firstly the Cartan-Killing form on the special orthogonal group $\mathbf{SO}(N)$ corresponds to the bilinear form $B(X, Y) = (N-2)\mathrm{tr}(XY)$ with $X, Y \in \mathfrak{so}(N)$. In virtue of the semisimplicity of $\mathbf{SO}(N)$ the latter leads to an inner product on the Lie algebra $\mathfrak{so}(N)$, here, taking the explicit form

$$(X,Y) = -\frac{B(X,Y)}{2(N-2)} = \frac{1}{2}\mathrm{tr}(XY^t) = \frac{1}{2}\mathrm{tr}(X^tY). \tag{6.2.1}$$

This inner product in turn results in a bi-variant metric on $\mathbf{SO}(N)$ that from now on is the choice of Riemannian metric. Specifically using left translations this gives the Adjoint invariant metric on $\mathbb{G} = \mathbf{SO}(N)$ defined via,

$$\rho(X_g, Y_g) = \frac{1}{2}\mathrm{tr}((g^{-1}X_g)^t g^{-1}Y_g) = \frac{1}{2}\mathrm{tr}(X^tY). \tag{6.2.2}$$

As the metric is bi-invariant the Riemannian exponential and the Lie exponential coincide (see, e.g., S. Helgason [40]) and so the geodesic $\gamma = \gamma(t)$ starting at $g \in \mathbb{G}$ in the direction $X_g \in T_g\mathbb{G}$ is given by $\gamma(t) = g\exp(tX)$ where $X \in \mathfrak{g}$ is such that $X_g = gX$. Now we fix the maximal torus $\mathbb{T}$ on $\mathbb{G}$ by setting, for even N,

$$\mathbb{T} = \Big\{ g \in \mathbf{SO}(2n) : g = \mathrm{diag}(\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n) \Big\}, \tag{6.2.3}$$

with the 2×2 rotation blocks $\mathcal{R}_j$ $(1 \le j \le n)$ in $\mathbf{SO}(2)$ given by $\mathcal{R}_j = \exp \mathcal{J}_j$:

$$\mathcal{R}_j = \mathcal{R}(a_j) = \begin{bmatrix} \cos a_j & -\sin a_j \\ \sin a_j & \cos a_j \end{bmatrix}, \qquad \mathcal{J}_j = \mathcal{J}(a_j) = \begin{bmatrix} 0 & -a_j \\ a_j & 0 \end{bmatrix},$$

$(a_j \in \mathbb{R})$ and the usual adjustment for odd N, namely, n, 2×2 block as above and a last 1×1 block consisting of entry 1, specifically,

$$\mathbb{T} = \Big\{ g \in \mathbf{SO}(2n+1) : g = \mathrm{diag}(\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n, 1) \Big\}. \tag{6.2.4}$$

The subalgebra $\mathfrak{t} \subset \mathfrak{g}$ corresponding to the maximal torus $\mathbb{T} \subset \mathbb{G}$ for even N is given by,

$$\mathfrak{t} = \Big\{ \xi \in \mathfrak{g} : \xi = \mathrm{diag}(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_n) \Big\}, \qquad N = 2n, \tag{6.2.5}$$

and for odd N by

$$\mathfrak{t} = \Big\{ \xi \in \mathfrak{g} : \xi = \mathrm{diag}(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_n, 0) \Big\}, \qquad N = 2n+1. \tag{6.2.6}$$

For the geodesic $\gamma(t) = \exp(tX)$ there exists $g \in \mathbf{SO}(N)$ so that $X = g\xi g^{-1}$ for suitable $\xi \in \mathfrak{t}$ and $\gamma(t) = g\exp(t\xi)g^{-1}$. It is therefore plain that any periodic geodesic at identity is *conjugate* to one sitting entirely on the maximal torus $\mathbb{T}$. Thus in considering the periodic geodesics of $\mathbb{G}$ we can merely focus on those confined to $\mathbb{T}$. Now let $\Lambda \subset \mathfrak{t}$ denote the lattice

$$\Lambda = \Big\{ \xi \in \mathfrak{t} : \exp(2\pi\xi) = \mathrm{I}_N \Big\}. \tag{6.2.7}$$

Then any closed geodesic on $\mathbb{T}$ is the Lie exponential $\gamma(t) = \exp(2\pi t\xi)$ for some $\xi \in \Lambda$ with $-\infty < t < \infty$. Note $\gamma(0) = \gamma(1) = \mathrm{I}_N$. Now let E_j (with $1 \le j \le n$) denote the block diagonal matrix

$$E_j = \mathrm{diag}(0, ..., 0, \mathscr{J}, 0, ..., 0), \qquad \mathscr{J} = \mathscr{J}(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{6.2.8}$$

Then recalling the inner product (6.2.1) it is seen that $(E_j : 1 \le j \le n)$ forms an orthonormal basis for $\mathfrak{t}$, hence, as any ξ in this subalgebra can be expressed as $\xi = \sum_{j=1}^n a_j E_j$, upon exponentiating we have,

$$\exp(2\pi\xi) = \exp\left(\sum_{j=1}^n 2\pi a_j E_j\right) = \prod_{j=1}^n \exp(2\pi a_j E_j).$$

As a result

$$\begin{aligned} \xi \in \Lambda \iff \exp(2\pi\xi) = \mathrm{I}_N \iff \exp(2\pi a_j E_j) = \mathrm{I}_N \\ \iff \begin{cases} \mathrm{diag}[\mathcal{R}(2\pi a_1), ..., \mathcal{R}(2\pi a_n)] = \mathrm{I}_N & N = 2n, \\ \mathrm{diag}[\mathcal{R}(2\pi a_1), ..., \mathcal{R}(2\pi a_n), 1] = \mathrm{I}_N & N = 2n+1, \end{cases} \\ \iff a_j \in \mathbb{Z} \qquad \text{for all } 1 \le j \le n. \end{aligned} \tag{6.2.9}$$

This therefore identifies Λ with $\mathbb{Z}^n$ which then via conjugation and translation describes all the periodic geodesics on $\mathbb{G} = \mathbf{SO}(N)$. Furthermore by conjugating Λ in $\mathfrak{so}(N)$ through $\mathbf{SO}(N)$ and using dimensional analysis it follows that every geodesic in $\mathbf{SO}(3)$ is periodic a conclusion that dramatically fails in $\mathbf{SO}(N)$ for $N \ge 4$. Now since the metric ρ on $\mathbf{SO}(N)$ is bi-invariant we have

$$|\dot\gamma(t)| = \left| g_1 g \frac{d}{dt} \exp(2\pi t\xi) g^{-1} \right| = \sqrt{2\pi^2 \mathrm{tr}(\xi^t \xi)} = 2\pi|\xi|. \tag{6.2.10}$$

Thus the length of the closed geodesic $\gamma = \gamma(t)$ is given by $l(\gamma) = 2\pi|\xi|$. Hence modulo translations and conjugations the number of closed geodesics whose length do not exceed $\sqrt{x} > 0$ can be expressed as

$$\mathscr{L}(x) = \sharp\left\{\xi \in \Lambda : |\xi| \le \frac{\sqrt{x}}{2\pi}\right\} = \sum_{y \in \mathbb{Z}^n} \chi_r(y), \tag{6.2.11}$$

where $r = \sqrt{x}/(2\pi)$ and χ_r is the characteristic function of the closed ball centred at origin with radius $r > 0$. Thus the geodesic counting function (in the sense described) connects to the Gauss circle problem and its higher dimensional analogues – a highly challenging and notoriously difficult problem in analytic number theory. Indeed $\mathscr{L}(x)$ counting the number of points in $\mathbb{Z}^n \cap \mathbb{B}_{\sqrt{x}/2\pi}$ has asymptotics given for suitable exponents δ, ζ (*see* the table below) by

$$\mathscr{L}(x) \sim \frac{\omega_n x^{\frac{n}{2}}}{(2\pi)^n} + O\left(x^\delta ln^\zeta x\right), \quad x \nearrow \infty. \tag{6.2.12}$$

Table 6.1: The best known exponents for the remainder term in $\mathscr{L}(x)$

	$n=2$	$n=3$	$n=4$	$n \geq 5$
(δ, ζ)	$(131/416, 0)$	$(21/32, 0)$	$(1, 2/3)$	$((n-2)/2, 0)$
Ref.	[46]	[38]	[80]	[32]

For $n \geq 5$ the δ in Table 6.1 is sharp whereas in the cases $n \leq 3$ finding the sharp δ is still an open problem. A conjecture of Hardy asserts that the sharp δ for $n = 2$ has the form $\delta = 1/4 + \varepsilon$ for all $\varepsilon > 0$ whilst in the case $n = 3$ the sharp δ is conjectured to be $\delta = 1/2 + \varepsilon$ for all $\varepsilon > 0$. (*See* the references for more.)

Next we denote by $\mathfrak{R}$ the set of roots, by Δ the corresponding root base and by F the set of fundamental weights. Due to the difference in the root structure of $\mathbf{SO}(N)$ when $N = 2n$ or $N = 2n+1$ we describe these two different cases separately.

- $\mathbf{SO}(2n)$:

$$\begin{aligned}
\mathfrak{R} &= \left\{ i(\pm E_k \pm E_l) : k > l \text{ with } 1 \leq l \leq n \right\}, \\
\Delta &= \left\{ i(E_j - E_{j+1}) : \text{ with } 1 \leq j \leq n-1 \right\} \bigcup \left\{ i(E_{n-1} + E_n) \right\}, \\
F &= \left\{ \mu_j = i \sum_{k=1}^{j} E_k : 1 \leq j \leq n-2 \right\} \bigcup \left\{ \mu_{n-1}, \mu_n = \frac{i}{2} \left(\sum_{k=1}^{n-1} E_k \mp E_n \right) \right\}.
\end{aligned}$$

The lattices of weights and analytic weights for $\mathbf{SO}(2n)$ are respectively given by,

$$\mathscr{P} = \text{span}_{\mathbb{Z}} F = \left\{ \sum_{i=1}^{n} b_i E_i : b \in \mathbb{Z}^n + \varepsilon(1, \dots, 1) \right\}, \quad \mathscr{A} = \left\{ \sum_{i=1}^{n} b_i E_i : b_i \in \mathbb{Z} \right\}.$$

Above ε is allowed to take the value of 0 or 1/2. Moreover, the set of analytic and dominant weights (the highest weights) is given by

$$\mathscr{A} \cap \mathscr{C}_+ = \left\{ \sum_{i=1}^{n} b_i E_i : b_i \in \mathbb{Z} \text{ and } b_1 \geq b_2 \geq \cdots \geq |b_n| \right\}. \tag{6.2.13}$$

Note that the $\mathscr{C}_+$ denotes the positive Weyl chamber corresponding to the choice of root base Δ.

- $\mathbf{SO}(2n+1)$:

$$\mathfrak{R}=\Big\{i(\pm E_l \pm E_k) : k>l \text{ with } 1\le l\le n\Big\}\bigcup\Big\{\pm iE_k : 1\le k\le n\Big\},$$
$$\Delta=\Big\{i(E_j-E_{j+1}) : 1\le j\le n-1\Big\}\bigcup\Big\{iE_n\Big\},$$
$$F=\Big\{\mu_j=i\sum_{k=1}^{j}E_k : 1\le j\le n-1\Big\}\bigcup\Big\{\mu_n=\frac{i}{2}\sum_{k=1}^{n}E_k\Big\}.$$

As in the case of $\mathbf{SO}(2n)$ we have that the weights and analytic weights are given by,

$$\mathscr{P}=\mathrm{span}_{\mathbb{Z}}F=\Big\{\sum_{i=1}^{n}b_iE_i : b\in\mathbb{Z}^n+\varepsilon(1,\dots,1)\Big\},\quad \mathscr{A}=\Big\{\sum_{i=1}^{n}b_iE_i : b_i\in\mathbb{Z}\Big\}.$$

Again $\varepsilon=0$ or $1/2$. However in this case we have the set of analytic and dominant weights given by

$$\mathscr{A}\cap\mathscr{C}_+=\Big\{\sum_{i=1}^{n}b_iE_i : b_i\in\mathbb{Z} \text{ and } b_1\ge b_2\ge\cdots\ge b_n\ge 0\Big\}. \tag{6.2.14}$$

Therefore in the $\mathbf{SO}(2n+1)$ case, in contrast to $\mathbf{SO}(2n)$, we have $b_n\in\mathbb{N}_0$.

It is well known that for a compact Lie group $\mathbb{G}$ equipped with a bi-invariant metric g the Laplace-Beltrami operator $-\Delta_g$ has spectrum $\Sigma=(\lambda_\mu)$ with

$$\lambda_\mu=(\mu+\rho,\mu+\rho)-(\rho,\rho)=||\mu+\rho||^2-||\rho||^2=(\mu,\mu)+2(\mu,\rho) \tag{6.2.15}$$

where ρ is the half-sum of positive roots and $\mu\in\mathscr{A}\cap\mathscr{C}_+$ (*cf.*, e.g., Knapp [51]). Moreover the multiplicity of the eigenvalue λ_μ is $\dim(\pi_\mu)^2$ where $\pi_\mu\in\hat{\mathbb{G}}$ (the unitary dual of $\mathbb{G}$) is the irreducible representation associated to $\mu\in\mathscr{A}\cap\mathscr{C}_+$, while $\dim(\pi_\mu)$ is given by Weyl's dimension formula. Restricting to $\mathbf{SO}(N)$ the eigenvalues of $-\Delta_g$ denoted $\Sigma=(\lambda_\omega : \omega\in\mathscr{A}\cap\mathscr{C}_+)$ are given by the explicit expression

$$\lambda_\omega=\begin{cases}\sum_{j=1}^{n}b_j(b_j+2n-2j), & \text{if } N=2n,\\ \sum_{j=1}^{n}b_j(b_j+2n+1-2j), & \text{if } N=2n+1,\end{cases} \tag{6.2.16}$$

where $b=(b_1,\dots,b_n)\in\mathbb{Z}^n$ and $b_1\ge b_2\ge\cdots\ge b_n\ge 0$ when $N=2n+1$ and $b_1\ge b_2\ge\cdots\ge|b_n|\ge 0$ when $N=2n$. Now to describe the multiplicities using

$$\dim(\pi_\omega)=\frac{\prod_{\alpha\in\mathfrak{R}^+}(\alpha,\omega+\rho)}{\prod_{\alpha\in\mathfrak{R}^+}(\alpha,\rho)}, \tag{6.2.17}$$

we first note that

$$\rho=\frac{1}{2}\sum_{\alpha\in\mathfrak{R}^+}\alpha=i\begin{cases}\sum_{j=1}^{n-1}(n-j)E_j, & \text{if } N=2n,\\ \sum_{j=1}^{n}(n-j+1/2)E_j, & \text{if } N=2n+1.\end{cases} \tag{6.2.18}$$

Therefore the multiplicity of the eigenvalue λ_ω is given by

$$m_n(x) = \dim(\pi_\omega)^2 = \frac{\prod_{i<l}(x_i^2 - x_l^2)^2}{\prod_{j<l}(a_j^2 - a_l^2)^2}, \qquad N = 2n, \tag{6.2.19}$$

$$m_n(x) = \frac{\prod_i x_i^2 \prod_{i<l}(x_i^2 - x_l^2)^2}{\prod_j a_j^2 \prod_{j<l}(a_j^2 - a_l^2)^2}, \qquad N = 2n+1. \tag{6.2.20}$$

Here $a_j = n-j$ when $N = 2n$ and $a_j = n-j+1/2$ when $N = 2n+1$ with $x = (x_1, \dots, x_n)$ given by,

$$x_j = b_j + \begin{cases} n-j, & \text{if } N = 2n, \\ n-j+1/2, & \text{if } N = 2n+1. \end{cases} \tag{6.2.21}$$

6.3 Counting lattice points with polynomial multiplicities

Let $\Gamma \subset \mathbb{R}^n$ be a lattice of full rank, that is, $\Gamma = \{\sum_{j=1}^n \ell_j v_j : \ell_j \in \mathbb{Z}\}$ where $v_1, ..., v_n$ is a fixed set of linearly independent vectors in $\mathbb{R}^n$. Assume $F = F(\lambda)$ is a homogenous polynomial of degree $d \geq 1$ on $\mathbb{R}^n$ assigning to each lattice point $\lambda \in \Gamma$ an associated *multiplicity* or weight $F(\lambda)$. The aim here is to describe the asymptotics of the weighted lattice point counting function

$$\mathscr{M}(R) = \sum_{\lambda \in \Gamma} F(\lambda)\chi_R(\lambda) = \sum_{\lambda \in \Gamma} F_R(\lambda). \tag{6.3.1}$$

Here χ_R denotes the characteristic function of the closed ball in $\mathbb{R}^n$ centred at the origin with radius $R > 0$ and $F_R = F\chi_R$. The approach is an adaptation of the classical argument in [42] based on smoothing out the sum via convolution with a mollifier and then using the Poisson summation formula. To this end we consider first the "mollified sum"

$$\begin{aligned} \mathscr{M}_\varepsilon(R) &= \sum_{\lambda \in \Gamma} [F_R \star \rho_\varepsilon](\lambda) = \mathrm{Vol}(\Gamma^\star) \sum_{\xi \in \Gamma^\star} \widehat{F_R}(\xi)\widehat{\rho_\varepsilon}(\xi) \\ &= \mathrm{Vol}(\Gamma^\star) \int_{\mathbb{R}^n} F_R(v)\, dv + \mathrm{Vol}(\Gamma^*) \sum_{\Gamma^\star \setminus \{0\}} \widehat{F_R}(\xi)\widehat{\rho_\varepsilon}(\xi), \end{aligned} \tag{6.3.2}$$

where $\Gamma^\star$ denotes the lattice dual to Γ. The focus will now be on the asymptotics of the second term on the right. Indeed since

$$\left| \sum_{\Gamma^* \setminus \{0\}} \widehat{F_R}(\xi)\widehat{\rho_\varepsilon}(\xi) \right| \leq \sum_{\Gamma^* \setminus \{0\}} |\widehat{F_R}(\xi)||\widehat{\rho_\varepsilon}(\xi)| \tag{6.3.3}$$

and by basic properties of the Fourier transform

$$\widehat{F_R}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} F_R(x)\, dx = R^{n+d}\widehat{F_1}(R\xi) \tag{6.3.4}$$

with $|\widehat{\rho_\varepsilon(\xi)}| = |\widehat{\rho}(\varepsilon\xi)| \leq c_k(1+\varepsilon|\xi|)^{-k}$ we can write

$$\left| \sum_{\Gamma^*\backslash\{0\}} \widehat{F_R}(\xi)\widehat{\rho_\varepsilon}(\xi) \right| \leq c_k R^{n+d} \sum_{\Gamma^*\backslash\{0\}} \frac{|\widehat{F_1}(R\xi)|}{(1+\varepsilon|\xi|)^k}. \tag{6.3.5}$$

So we now need to describe the behaviour of $\widehat{F_1}(R\xi)$ for large R. Towards this end and in virtue of F being homogeneous we proceed by expressing F as

$$F(x) = |x|^d P\left(\frac{x}{|x|}\right) = |x|^d \sum_{k=1}^{[\frac{d}{2}]} P_k(\theta), \qquad \theta = x/|x|, \tag{6.3.6}$$

where $P = P(\theta)$ is the restriction to the sphere $\mathbb{S}^{n-1}$ of F and $P_k = P_k(\theta)$ is a spherical harmonic of degree $d-2k$ (*see*, e.g., L. Grafakos [37] or Stein & Weiss [72]). Next let us denote

$$F_1(x) = \sum_{k=1}^{[\frac{d}{2}]} F_{k,1}(x), \qquad F_{k,1}(x) = |x|^{2k} P_k(x)\chi_1(x), \tag{6.3.7}$$

where each P_k is a solid spherical harmonic on $\mathbb{R}^n$ of degree $d-2k$. Clearly by the linearity of the Fourier transform we can write

$$\widehat{F_1}(\xi) = \sum_{k=1}^{[\frac{d}{2}]} \widehat{F_{k,1}}(\xi), \qquad \widehat{F_{k,1}}(\xi) = \widehat{f_{k,1}}(|\xi|)P_k(\xi).$$

We now momentarily focus on the quantity $\widehat{f_{k,1}}(|\xi|)$

$$\begin{aligned}\widehat{f_{k,1}}(|\xi|) &= \frac{2\pi i^{2k-d}}{|\xi|^{(n+2d-4k-2)/2}} \int_0^1 s^{(n+2d)/2} J_{(n+2d-4k-2)/2}(2\pi|\xi|s)\, ds \\ &= \frac{(2\pi)^{-(n+2d)/2} i^{2k-d}}{|\xi|^{(n+2d-2k)}} \int_0^{2\pi|\xi|} s^{(n+2d)/2} J_{(n+2d-4k-2)/2}(s)\, ds.\end{aligned}$$

By invoking an estimate for the weighted integral of Bessel functions that the reader can find in the Appendix (*see* Proposition 6.6.1) we have

$$\int_0^{2\pi|\xi|} s^{(n+2d)/2} J_{(n+2d-4k-2)/2}(s) ds \leq C_k \left(2\pi|\xi|\right)^{(n+2d-1)/2}$$

when $|\xi| > M_k$ for some $M_k \in \mathbf{R}$. Therefore for $|\xi| > M_k$ we can write

$$|\widehat{f_{k,1}}(|\xi|)| \leq C_k |\xi|^{-(\frac{n+1}{2}+d-2k)}.$$

This in turn means that,

$$|\widehat{F_1}(R\xi)| = \sum_{k=1}^{[\frac{d}{2}]} |R\xi|^{d-2k} |\widehat{f_{k,1}}(|R\xi|)| \left| P_k\left(\frac{\xi}{|\xi|}\right)\right| \leq c|R\xi|^{-(n+1)/2} \tag{6.3.8}$$

for $|\xi| > \max_k(M_k)$. Therefore returning to the remainder term we get that for large enough R,

$$
\begin{aligned}
\left|\sum_{\Gamma^*\setminus\{0\}} \widehat{F_R}(\xi)\widehat{\rho_\varepsilon}(\xi)\right| &\le c \sum_{\Gamma^*\setminus\{0\}} \frac{R^{d+(n-1)/2}}{|\xi|^{(n+1)/2}\,(1+\varepsilon|\xi|)^k} \\
&\le c \int_{|\xi|\ge 1} \frac{R^{d+(n-1)/2}\,d\xi}{|\xi|^{(n+1)/2}\,(1+\varepsilon|\xi|)^k} \\
&\le c\varepsilon^{-\frac{n-1}{2}} \int_{\mathbf{R}^n} \frac{R^{d+(n-1)/2}\,d\xi}{|\xi|^{(n+1)/2}\,(1+|\xi|)^k} \\
&\le c\frac{R^{d+(n-1)/2}}{\varepsilon^{\frac{n-1}{2}}}. \qquad (6.3.9)
\end{aligned}
$$

As a result we can conclude that the "mollified sum" from (6.3.2) has the asymptotic behaviour

$$
\begin{aligned}
\mathscr{M}_\varepsilon(R) &= \mathrm{Vol}(\Gamma^*) \int_{\mathbb{R}^n} F_R(x)\,dx + O\left(R^{d+\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}\right) \\
&= \mathrm{Vol}(\Gamma^*)R^{d+n} \int_{\mathbb{R}^n} F_1(x)\,dx + O\left(R^{d+\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}\right). \qquad (6.3.10)
\end{aligned}
$$

We next compare the mollified counting function with the original one. Towards this end we first observe that for $y \in \mathbb{B}_R$ there exists some $z \in \mathbb{B}_{R+\epsilon}$ such that $F_{R+\epsilon} \star \rho_\epsilon(y) = F_{R+\epsilon}(z)$. Thus by combining the above

$$
|F_R(y) - F_{R+\varepsilon} \star \rho_\varepsilon(y)| = |F_R(y) - F_{R+\varepsilon}(z)| \le 2\varepsilon \max_{x\in\mathbb{B}_\varepsilon(y)} |\nabla F(x)| \le C\varepsilon R^{d-1},
$$

Therefore it follows that for each $y \in \mathbb{B}_R(0)$ we have,

$$
F_{R+\varepsilon} \star \rho_\varepsilon(y) - C\varepsilon R^{d-1} \le F_R(y) \le F_{R+\varepsilon} \star \rho_\varepsilon(y) + C\epsilon R^{d-1}
$$

thus giving

$$
\mathscr{M}_\varepsilon(R+\varepsilon) - C\varepsilon R^{d-1} \sum_{\lambda\in\Gamma} \chi_R(\lambda) \le \mathscr{M}(R) \le \mathscr{M}_\varepsilon(R+\varepsilon) + C\epsilon R^{d-1} \sum_{\lambda\in\Gamma} \chi_R(\lambda). \qquad (6.3.11)
$$

Next referring to the original lattice Γ we can define with the aid of the basis vectors $v_1, ..., v_n$ another lattice

$$
\Omega = \left\{\omega = \sum_{j=1}^{n} \ell_j \frac{a v_j}{||v_j||} : \ell_j \in \mathbb{Z}\right\} \qquad (6.3.12)
$$

where $a = \min_j\{\|v_j\|\} > 0$. Then using the bound

$$
\sum_{\Gamma} \chi_R \le \sum_{\Omega} \chi_R = \sum_{\mathbb{Z}^n} \chi_{\frac{R}{a}} \le \frac{R^n}{a^n}, \qquad (6.3.13)
$$

we can rewrite (6.3.11) as

$$
\mathscr{M}_\varepsilon(R+\varepsilon) - C\varepsilon R^{d+n-1} \le \mathscr{M}(R) \le \mathscr{M}_\varepsilon(R+\varepsilon) + C\epsilon R^{d+n-1}.
$$

Therefore the previously obtained bounds for $\mathscr{M}_\varepsilon$ result in

$$\mathscr{M}(R) = \left(\mathrm{Vol}(\Gamma^*) \int_{\mathbb{R}^n} F_1(x)\, dx\right) R^{d+n} + O\left(R^{d+\frac{n-1}{2}} \varepsilon^{-\frac{n-1}{2}} + \varepsilon R^{d+n-1}\right)$$

as $R \nearrow \infty$. Noting that the remainder term is optimised when $\varepsilon = R^{-\frac{n-1}{n+1}}$ leads to the following conclusion.

Theorem 6.3.1. Let F be a homogeneous polynomial of degree $d \geq 1$ on $\mathbb{R}^n$ and let $\Gamma \subset \mathbb{R}^n$ be a lattice of full rank. Consider the weighted counting function $\mathscr{M} = \mathscr{M}(R)$ defined for $R > 0$ by (6.3.1). Then

$$\mathscr{M}(R) = \left(\mathrm{Vol}(\Gamma^*) \int_{\mathbb{B}_1^n} F(x)\, dx\right) R^{d+n} + O\left(R^{d+n-\frac{2}{n+1}}\right), \qquad R \nearrow \infty. \tag{6.3.14}$$

Note that repeating the above proof for the shifted counting function $\mathscr{M}^h(R) = \sum_{\lambda \in \Gamma} F_R(\lambda + h)$ with $\mathscr{M}_\varepsilon^h(R) = \sum_{\lambda\in\Gamma}[F_R * \rho_\varepsilon](\lambda + h)$ results in the exact same asymptotics (6.3.14) for $\mathscr{M}^h$. This is a consequence of the identities $\widehat{f(\cdot + h)}(\xi) = \widehat{f}(\xi) e^{2\pi i h\cdot\xi}$, $|\widehat{f(\cdot + h)}(\xi)| = |\widehat{f}(\xi)|$. We use this remark later on.

6.4 Improved asymptotics for $\mathscr{N}(\lambda; \mathbb{G})$ when $\mathbb{G} = \mathbf{SO}(N)$, $\mathbf{SU}(N)$, $\mathbf{U}(N)$ and $\mathbf{Spin}(N)$

This section is devoted to the analysis of the asymptotics of the spectral counting function $\mathscr{N}(\lambda; \mathbb{G})$ as $\lambda \nearrow \infty$ when $\mathbb{G}$ is one of the special orthogonal or unitary groups in the title. Here, the calculations in light of what has been obtained so far is explicit and the main question is the behaviour of the remainder term and whether it agrees with the Avakumovic-Hörmander sharp form or if there is an improvement. Notice that $\mathbf{SO}(2) \cong \mathbb{S}^1$ and $\mathbf{SO}(3) \cong \mathbb{S}^3/\{\pm 1\} \cong \mathbf{P}(\mathbb{R}^3)$, the real projective space, and so in view of the periodicity of the geodesic flow (or direct calculations) we do not expect any improvements. However remarkably things change sharply as soon as we pass to the higher dimensional cases $\mathbf{SO}(N)$ (with $N \geq 4$). Indeed from earlier discussions we know, using (6.2.16)-(6.2.19), that the spectral counting function for $\mathbf{SO}(N)$ is given by (below we shall be using the notation of $\mathscr{A}_\rho = \mathscr{A} + \rho$.)

$$\mathscr{N}(\lambda) = \sum_{\omega \in \mathscr{A} \cap \mathscr{C}_+} m_n(\omega + \rho) \chi_R(\omega + \rho) = \sum_{x \in \mathscr{A}_\rho \cap \mathring{\mathscr{C}}_+} m_n(x) \chi_R(x), \tag{6.4.1}$$

where χ_R is the characteristic function of the closed ball with $R = \sqrt{\lambda + \|\rho\|^2}$ centred at the origin and m_n is the multiplicity function that is explicitly by (6.2.19). In (6.4.1) we have also let $x = \omega + \rho$ and used the fact that $x \in \mathring{\mathscr{C}}_+$ for $\omega \in \mathscr{A} \cap \mathscr{C}_+$. An easy inspection show that on $\partial\mathscr{C}_+$ we have $m_n(x) = 0$ therefore we can rewrite $\mathscr{N}(\lambda)$ as

$$\mathscr{N}(\lambda) = \sum_{x \in \mathscr{A}_\rho \cap \mathscr{C}_+} m_n(x)\chi_R(x). \tag{6.4.2}$$

Notice that the multiplicity function m_n is invariant under any permutation of $(x_1, \cdots, x_n)$ in either case. In addition m_n is also invariant under any change in sign of $n-1$ of the x_i's when $N = 2n$ and invariant under any change of sign of all the x_i's when $N = 2n+1$. Thus m_n is invariant under the Weyl group W given by

$$W = \begin{cases} \mathbb{Z}_2^{n-1} \rtimes S_n, & \text{if } N = 2n, \\ \mathbb{Z}_2^n \rtimes S_n, & \text{if } N = 2n+1. \end{cases} \tag{6.4.3}$$

We now take advantage of the action of the Weyl group on the set of weights $\mathscr{A}_\rho \cap \mathscr{C}_+$ to extend $\mathscr{N}(\lambda)$ to the full set of weights $\mathscr{A}_\rho$. Note that the Weyl group W maps $\mathscr{A}_\rho$ to itself since for each $w \in W$ we have that $w \cdot \rho = \rho - \alpha$ for some $\alpha \in \mathfrak{R} \subset \mathscr{A}$ (*see* Proposition 20.15 in [18]). Then as $W \cdot \mathscr{A} = \mathscr{A}$ (*see* Proposition 20.16 in [18]) we clearly have that for any $w \in W$, $w \cdot (\mu + \rho) \in \mathscr{A}_\rho$. Indeed as the Weyl group acts simply transitively on the interior of the Weyl chambers (which means that the interior of any Weyl chamber is mapped onto the interior of any other chamber in a bijective manner) we can write

$$\mathscr{N}(\lambda) = \frac{1}{|W|}\sum_{x \in \mathscr{A}_\rho} m_n(x)\chi_R(x) = \frac{1}{|W|}\sum_{\omega \in \mathscr{A}} m_n(\omega+\rho)\chi_R(\omega+\rho) = \mathscr{M}^\rho(R).$$

Now m_n is a homogeneous polynomial of degree $2l = d - n$ where $d = \dim[\mathbf{SO}(N)]$ and $n = \mathrm{Rank}[\mathbf{SO}(N)]$. Then since $\mathscr{A}$ can be identified with $\mathbb{Z}^n$ we have that $\mathrm{Vol}(\mathscr{A}^*) = 1$ and then from Theorem 6.3.1 we deduce that

$$\mathscr{M}^\rho(R) = \frac{R^d}{|W|}\int_{\mathbf{R}^n} m_n(x)\chi_1(x)dx + O\left(R^{d-1-\frac{n-1}{n+1}}\right),$$

as $R \nearrow \infty$. Hence in view of $\mathscr{N}(\lambda) = \mathscr{M}(R)$, when $R = \sqrt{\lambda + \|\rho\|^2}$, we obtain

$$\mathscr{N}(\lambda) = \frac{\lambda^{d/2}}{|W|}\int_{\mathbf{R}^n} m_n(x)\chi_1(x)dx + O\left(\lambda^{\frac{d-1}{2} - \frac{n-1}{2(n+1)}}\right).$$

The leading term can be evaluated to be

$$\frac{\lambda^{d/2}}{|W|}\int_{\mathbb{B}_1} m_n(x)dx = \frac{\omega_d \mathrm{Vol}_g(\mathbf{SO}(N))}{(2\pi)^d}\lambda^{d/2}.$$

Subsequently it follows that

$$\mathscr{N}(\lambda) = \frac{\omega_d \mathrm{Vol}_g(\mathbf{SO}(N))}{(2\pi)^d}\lambda^{\frac{d}{2}} + O(\lambda^\alpha), \quad \lambda \nearrow \infty, \tag{6.4.4}$$

where the exponent α in the remainder term, by making use of $d = (N^2 - N)/2$, is seen to be

$$\alpha = \frac{1}{2(n+1)} \begin{cases} 2n^3 + n^2 - 3n & \text{if } N = 2n, \\ 2n^3 + 3n^2 - n & \text{if } N = 2n+1. \end{cases} \tag{6.4.5}$$

This in particular confirms that the remainder term in Weyl's law is not sharp for the compact Lie group $\mathbf{SO}(N)$. In summary we have proved the following result.

Theorem 6.4.1. The spectral counting function $\mathscr{N} = \mathscr{N}(\lambda; \mathbf{SO}(N)$ of the Laplace-Beltrami operator with $n = \text{rank}[\mathbf{SO}(N)] \geq 2$ has the asymptotics $(\lambda \nearrow \infty)$

$$\mathscr{N}[\lambda; \mathbf{SO}(N)] = \frac{\text{Vol}(\mathbf{SO}(N))\omega_d}{(2\pi)^d} \lambda^{\frac{d}{2}} + O(\lambda^\alpha), \tag{6.4.6}$$

with $d = \dim(\mathbf{SO}(N))$ and α given by (6.4.5).

Table 6.2: $\mathbf{SO}(N)$ and $\mathbf{Spin}(N)$

$\mathbb{G}$	$\mathbf{SO}(N)$	$\mathbf{Spin}(N)$
$n = \text{rank}(\mathbb{G})$	$[N/2]$	$[N/2]$
$d = \dim(\mathbb{G})$	$N(N-1)/2$	$N(N-1)/2$
$Q = \prod_{\alpha \in R^+}(\alpha, \rho)$	$(2^{-n}N!!)^{N-2n} \prod_{j=1}^{n-1} j! \prod_{j<k}(N-j-k)$	
$\text{Vol}(\mathbb{G}) \times Q$	$(2\pi)^{N(N-1)/4+n/2}$	$2(2\pi)^{N(N-1)/4+n/2}$

We now present the analogous analysis and result for the unitary and special unitary groups $\mathbf{U}(N)$ and $\mathbf{SU}(N)$ respectively. Firstly note that the spectrum of the Laplace-Beltrami on $\mathbf{U}(N)$ and $\mathbf{SU}(N)$ is given by the following

$$\lambda_\omega = \sum_{j=1}^{N} \left[(b_j - j + (N+1)/2)^2 - ((N+1)/2 - j)^2 \right], \tag{6.4.7}$$

where $b_1 \geq b_2 \geq \cdots \geq b_N$ with $b_j \in \mathbb{Z}$ for $\mathbf{U}(N)$ or $b_j \in \mathbb{Z} + b_N$ for $1 \leq j \leq n-1$ and $b_N \in \mathbb{Z}/N$ whilst (6.4.9) holds for $\mathbf{SU}(N)$. The multiplicity of these eigenvalues is given by,

$$m_N(x) = \prod_{1 \leq j < k \leq N} \frac{(x_j - x_k)^2}{(k-j)^2}, \tag{6.4.8}$$

where $x = (x_1, \ldots, x_N)$ and $x_j = b_j - j + (N+1)/2$. Note that in the case of $\mathbf{SU}(N)$ the

$$b_N = -\sum_{j=1}^{N-1} b_j, \tag{6.4.9}$$

which therefore means that the eigenvalues and corresponding multiplicity function only depend on $b_1, \dots, b_{N-1}$ and $x_1, \dots, x_{N-1}$ respectively. Now following the arguments of $\mathbf{SO}(N)$ we can prove the following (the proof of which we shall omit due to the similarity with $\mathbf{SO}(N)$).

Theorem 6.4.2. The spectral counting function $\mathscr{N}(\lambda) = \mathscr{N}(\lambda; \mathbb{G})$ of the Laplace-Beltrami operator on the unitary group $\mathbb{G} = \mathbf{U}(N)$ with $N \geq 2$ and $d = \dim(\mathbf{U}(N)) = N^2$ has the asymptotics ($\lambda \nearrow \infty$)

$$\mathscr{N}[\lambda; \mathbf{U}(N)] = \frac{\mathrm{Vol}(\mathbf{U}(N))\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{N(N+2)(N-1)}{2(N+1)}}\right). \tag{6.4.10}$$

Likewise in the case of the special unitary group $\mathbb{G} = \mathbf{SU}(N)$ with $N \geq 2$ and $d = \dim(\mathbf{SU}(N)) = N^2 - 1$ we have that

$$\mathscr{N}[\lambda; \mathbf{SU}(N)] = \frac{\mathrm{Vol}(\mathbf{SU}(N))\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{N^3-3N+2}{2N}}\right). \tag{6.4.11}$$

Note that the metric is the one arising from the inner product $(X, Y) = \mathrm{tr}\,(X^\star Y)$ and is bi-invariant. By inspection for $\mathbf{U}(N)$ when $N \geq 2$ and for $\mathbf{SU}(N)$ when $N \geq 3$ the remainder term in Weyl's law (6.1.2) is not sharp whilst evidently outside this range the geodesic flow on the group is periodic.

Table 6.3: $\mathbf{SU}(N)$ and $\mathbf{U}(N)$

$\mathbb{G}$	$\mathbf{SU}(N)$	$\mathbf{U}(N)$
$n = \mathrm{rank}(\mathbb{G})$	$N-1$	N
$d = \dim(\mathbb{G})$	N^2-1	N^2
$Q = \prod_{\alpha \in R^+}(\alpha, \rho)$	$\prod_{j=1}^{N-1} j!$	$\prod_{j=1}^{N-1} j!$
$\mathrm{Vol}(\mathbb{G}) \times Q$	$N(2\pi)^{(N+2)(N-1)/2}$	$(2\pi)^{N(N+1)/2}$

Let us end the section by studying the asymptotics of $\mathscr{N}(\lambda; \mathbb{G})$ for when $\mathbb{G} = \mathbf{Spin}(N)$ is the universal cover of $\mathbf{SO}(N)$. In virtue of $\pi_1[\mathbf{Spin}(N)] \cong 0$ there is a one-to-one correspondence between the [complex] irreducible representations of $\mathbf{Spin}(N)$ and those of its Lie algebra $\mathfrak{spin}(N)$. Moreover as $\mathbf{Spin}(N)$ is a double cover of $\mathbf{SO}(N)$ the irreducible representations of the latter are only "half" the total of the former and hence of the Lie algebras $\mathfrak{spin}(N) \cong \mathfrak{so}(N)$. Despite this we have the following conclusion for $\mathbf{Spin}(N)$ based on what was obtained previously for $\mathbf{SO}(N)$.

Theorem 6.4.3. Consider the universal covering group $\mathbb{G} = \mathbf{Spin}(N)$ of $\mathbf{SO}(N)$ with $N \geq 3$. Then the spectral counting function $\mathscr{N} = \mathscr{N}(\lambda; \mathbb{G})$ of the Laplace-Beltrami operator has the asymptotics

$$\mathscr{N}(\lambda;\mathbb{G}) = \frac{\omega_d \mathrm{Vol}_g(\mathbb{G})}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\alpha}\right), \qquad \lambda \nearrow \infty, \tag{6.4.12}$$

where $d = \dim \mathbf{Spin}(N) = \dim \mathbf{SO}(N) = N(N-1)/2$ and α is given by (6.4.5).

Proof. As eluded to in the discussion prior to the theorem in virtue of $\mathbb{G} = \mathbf{Spin}(N)$ being the universal covering group of $\mathbf{SO}(N)$ the two share the same root system, and the set of analytically dominant weights $\mathscr{A} \cap \mathscr{C}_+$ for $\mathbf{Spin}(N)$ are,

$$\mathscr{A} \cap \mathscr{C}_+ = \left\{ \xi + \varepsilon(1,\cdots,1) : \xi \in \mathscr{A}_{\mathbf{SO}(N)} \cap \mathscr{C}_+,\ \varepsilon = 0 \text{ or } \varepsilon = \frac{1}{2} \right\} = \mathscr{P}_{\mathbf{SO}(N)} \cap \mathscr{C}_+. \tag{6.4.13}$$

The root systems of $\mathbf{Spin}(N)$ and $\mathbf{SO}(N)$ being the same implies that the multiplicity function m_n for the two are the same homogenous polynomial. In particular for each $y \in \mathscr{A} \cap \mathscr{C}_+$ we can write $y = x + \varepsilon(1,\ldots,1)$ for some $x \in \mathscr{A}_{\mathbf{SO}(N)} \cap \mathscr{C}_+$ Therefore the counting function of $\mathbf{Spin}(N)$ relates to the counting function of $\mathbf{SO}(N)$ by a summation, specifically,

$$\begin{aligned} \mathscr{M}_{\mathbf{Spin}(N)}(R) &= \mathscr{M}_{\mathbf{SO}(N)}(R) + \mathscr{M}^{1/2}_{\mathbf{SO}(N)}(R) \\ &= 2\frac{\omega_d \mathrm{Vol}_g(\mathbf{SO}(N))}{(2\pi)^d} R^d + O\left(R^{d-1-\frac{n-1}{n+1}}\right). \end{aligned} \tag{6.4.14}$$

Above $\mathscr{M}^{1/2}_{\mathbf{SO}(N)}(R)$ denotes the counting function over the shifted lattice $\mathbb{Z}^n + (1/2,\ldots,1/2)$.

Now we reach the desired conclusion by invoking the relation $R = \sqrt{\lambda + \|\rho\|^2}$ and hence obtaining

$$\mathscr{N}(\lambda) = 2\frac{\omega_d \mathrm{Vol}_g(\mathbf{SO}(N))}{(2\pi)^d}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{d-1}{2} - \frac{1}{2}\frac{n-1}{n+1}}\right) \tag{6.4.15}$$

and making use of the relation $2\mathrm{Vol}(\mathbf{SO}(N)) = \mathrm{Vol}(\mathbf{Spin}(N))$. $\square$

Regarding the last relation in the above proof we note that the volume of any compact Lie group $\mathbb{G}$ is given by

$$\mathrm{Vol}(\mathbb{G}) = \frac{(2\pi)^{n+l}}{\mathrm{Vol}(\mathscr{A}) \times Q}, \qquad Q = \prod_{\alpha \in \mathfrak{R}^+} (\alpha, \rho), \tag{6.4.16}$$

where $\mathrm{Vol}(\mathscr{A})$ is the volume of the fundamental domain in the lattice of analytical weights. Then as $\mathbf{Spin}(N)$ and $\mathbf{SO}(N)$ share the same root system and $\mathrm{rank}(\mathbb{G}) = n$ the above claim follows upon noting that $\mathrm{Vol}(\mathscr{A}_{\mathbf{SO}(N)}) = 1$ and $\mathrm{Vol}(\mathscr{A}_{\mathbf{Spin}(N)}) = 2^{-1}$, which can be seen by calculating the $|\det \mathbf{F}|$ where each column of $\mathbf{F}$ is fundamental weight in F. The specific form of the fundamental weights means $\mathbf{F}$ is either a triangular matrix or triangular apart from 2×2 block along diagonal.

6.5 Weyl's Law for the Orthogonal and Unitary groups: A sharp result

In this final section we present a result containing sharp asymptotics of the remainder term for the spectral counting function of the orthogonal and unitary groups as in the previous sections. Here we assume for technical reasons that the rank of the group is strictly greater than four.

Theorem 6.5.1. Let $\mathbb{G}$ denote one of the unitary, orthogonal or spinor groups as above. Then provided that $n = \mathrm{rank}(\mathbb{G}) \geq 5$ we have

$$\mathscr{N}(\lambda;\mathbb{G}) = \frac{\omega_d \mathrm{Vol}(\mathbb{G})}{(2\pi)^d}\lambda^{\frac{d}{2}} + O(\lambda^{\frac{d-2}{2}+\beta}), \qquad \lambda \nearrow \infty, \tag{6.5.1}$$

where as before $d = \dim(\mathbb{G})$ and β is given by

$$\beta = \begin{cases} 2 - \frac{n}{4} - \alpha & \text{if } n = 5,6,7, \\ 0 & \text{if } n \geq 8 \end{cases},$$

with $\alpha = 1/4 - \varepsilon$ for any $\varepsilon > 0$.

The principle idea of the proof is to approximate the counting function $\mathscr{N}(\lambda)$ by an alternative one with radial weight which is easier to tame. Recall that the multiplicity functions, given earlier in the paper, are homogeneous polynomials of even degree and as such can be written as $m_n(x) = |x|^{2m}P(x/|x|)$. This homogeneity permits the forthcoming description of $\mathscr{N}(\lambda)$ where $n = \mathrm{rank}(\mathbb{G})$ and $r_n(k) = |\{\omega \in \mathbb{Z}^n : |\omega|^2 = k\}|$. Indeed

$$\mathscr{N}(\lambda) = \frac{1}{|W|}\sum_{k=1}^{R^2} k^m \sum_{\theta_j \in H_k} P(\theta_j), \tag{6.5.2}$$

where $H_k = \{\theta_j = x_j/|x_j| : x_j \in \mathbb{Z}^n \text{ and } |x_j|^2 = k\}$. The form (6.5.2) is suggestive towards a natural approximation by

$$\mathscr{E}(R) = \sum_{\omega\in\mathbb{Z}^n} |\omega|^{2m}\chi_R(\omega) = \sum_{k=1}^{R^2} k^m r_n(k). \tag{6.5.3}$$

It is clear that any sensible approximation of $\mathscr{N}(\lambda)$ via $\mathscr{E}(R)$ necessitates that the inner sum in (6.5.2) converges to $\kappa r_n(k)$ where $\kappa = |\mathbb{S}^{n-1}|^{-1}\int_{\mathbb{S}^{n-1}} P(\theta)\, d\mathcal{H}^{n-1}(\theta)$. Consequently

$$|\mathscr{N}(\lambda) - \kappa|W|^{-1}\mathscr{E}(R)| \leq \frac{1}{|W|}\sum_{k=0}^{R^2} k^m \left|\sum_{\theta_j\in H_k} P(\theta_j) - \kappa r_n(k)\right|. \tag{6.5.4}$$

Let us suppose for the time being that,

$$\left|\sum_{\theta_j\in H_k} P(\theta_j) - \kappa r_n(k)\right| = O(k^{\frac{n}{4}-\alpha}), \tag{6.5.5}$$

where we shall define α explicitly later. Then,

$$|\mathscr{N}(\lambda) - \kappa|W|^{-1}\mathscr{E}(R)| = O\left(R^{2m+\frac{n+4}{2}-2\alpha}\right). \tag{6.5.6}$$

Therefore once we have a result for $\mathscr{E}(R)$ we shall obtain asymptotics for $\mathscr{N}$ as a simple repercussion of the above approximation estimate and the correct choice of α.

Theorem 6.5.2. Let $\mathscr{E} = \mathscr{E}(R)$ be as in (6.5.3). Then provided $n \geq 5$ we have the asymptotics

$$\mathscr{E}(R) = \frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{2m+n} R^{2m+n} + O(R^{2m+n-2}), \qquad R \nearrow \infty. \tag{6.5.7}$$

Proof. The proof of this result is an adaptation of the classical lattice point counting argument with constant weight, i.e., $m = 0$. Indeed the explicit form of $r_4(k)$, i.e., the Jacobi sum of four square formula, gives a weaker result for $\mathscr{E}(R)$ on $\mathbb{Z}^4$. More precisely,

$$\begin{aligned}\mathscr{E}_4(R) = &\int_{\mathbb{B}_R} |x|^{2m}\, dx + \\ &\frac{16/R^2}{\mathrm{Vol}(\mathbb{S}^3)} \int_{\mathbb{B}_R} |x|^{2m} dx \cdot \left[D(R^2) - D(R^2/4)\right] + O(R^{2m+2}),\end{aligned} \tag{6.5.8}$$

where $D(t) = \sum_{k\leq t} k^{-1}\psi(t/k)$ with $\psi(t/k) = t/k - [t/k] - 1/2$. To prove (6.5.8) one can firstly show that (*cf.* [32] pp. 34-5)

$$\mathscr{E}_4(R) = 8S_m(R^2) - 4^m \times 32\, S_m\left(\frac{R^2}{4}\right), \tag{6.5.9}$$

where $S_m(t) = \sum_{k=1}^{t} k^m\sigma(k)$ and $\sigma(k)$ is the classical divisor function, i.e., $\sigma(k) = \sum_{d|k} d$. In [32] the identity (6.5.9) is proved when $m = 0$. However the case $m > 0$ is similar modulo suitable adjustments to essentially account for $m \neq 0$. Therefore to avoid repetition with an existing text we shall simply refer the reader to [32]. Now note that any pair of integers (d, j) such that $1 \leq d \leq t$ and $1 \leq j \leq [t/d] = t/d - \psi(t/d) - 1/2 = \zeta(d, t)$ are divisors of $d \cdot j$ where $1 \leq d \cdot j \leq t$. Hence,

$$S_m(t) = \sum_{k\leq t} k^m \sum_{d|k} d = \sum_{k\leq t} \sum_{j\leq\zeta(k,t)} (jk)^m j = \sum_{k\leq t} \frac{k^m}{m+2} \sum_{j=0}^{m+2} c_{m,j}\zeta(k,n)^{m+2-j},$$

where the last equality comes from Faulhaber's formula with $c_{m,j} = (-1)^j\binom{m+2}{j}B_j$ and B_j are the Bernoulli numbers. Moreover,

$$\zeta(k,t)^m = \left(\frac{t}{k}\right)^m - \left(\frac{t}{k}\right)^{m-1}\left(\psi(\frac{t}{k}) + 1/2\right) + O\left(\frac{t^{m-2}}{k^{m-2}}\right). \tag{6.5.10}$$

As a result,

$$S_m(t) = \sum_{k\le t} \frac{k^m}{m+2} \sum_{j=0}^{m+2} c_{m,j} \left(\frac{t}{k}\right)^{m+1-j} \zeta(k,t) + O(t^{m+1})$$
$$= \frac{t^m}{m+2} \sum_{k\le t} \left(\frac{t}{k} + c_{m,1}\right) \zeta(k,t) + O(t^{m+1}). \tag{6.5.11}$$

Now $\sum_{k\le t} t/k\zeta(k,t) = t^2 \sum_{k\le t} k^{-2} - tD(t) - t/2 \sum_{k\le t} k^{-1}$ and $\sum_{k\le t} \zeta(k,t) = t\sum_{k\le t} k^{-1} + O(t)$ which permits (6.5.11) to be rewritten as (with $c = c_{m,1} - 2^{-1}$)

$$S_m(t) = \frac{t^{m+2}}{m+2} \sum_{k\le t} \frac{1}{k^2} + \frac{t^{m+1}}{m+2} \left[c \sum_{k\le t} k^{-1} - D(t) \right] + O(t^{m+1}). \tag{6.5.12}$$

Next substituting (6.5.12) into (6.5.9) with $\sum_{k\le R^2} k^{-1} - \sum_{k\le R^2/4} k^{-1} = O(1)$ and using $1 + \cdots + 1/N^2 = \pi^2/6 + O(N^{-1})$ gives

$$\mathscr{E}_4(R) = \frac{R^{2m+4}\pi^2}{m+2} - \frac{8R^{2m+2}}{m+2} \left[D(R^2) - D(R^2/4)\right] + O(R^{2m+2}). \tag{6.5.13}$$

Thus we have proved (6.5.8) upon identifying the coefficients in (6.5.13) with the integrals given in (6.5.8). Now (6.5.7) is obtained by firstly writing,

$$\mathscr{E}_5(R) = \sum_{\omega\in\mathbb{Z}^5} |\omega|^{2m} \chi_R(\omega) = \sum_{j=-R}^{R} \mathscr{E}_4^{f_j}(\sqrt{R^2 - j^2}). \tag{6.5.14}$$

Here $\mathscr{E}_4^{f_j}(t)$ denotes the counting function on $\mathbb{Z}^4$ with weight $f(\omega, j) = (|\omega|^2 + j^2)^m$ which is a sum of radial weights. Hence (6.5.8) in (6.5.14) produces,

$$\mathscr{E}_5(R) = \sum_{j=-R}^{R} \int_{\mathbb{B}_{r(R^2,j)}} f(x,j)\,dx - 16\left[H(R^2) - L(R^2)\right] + O(R^{2m+3}), \tag{6.5.15}$$

where we have defined $r(t,j) = \sqrt{t - j^2}$ and (below the remainder term accounts for $j = 0$ which is $t^{m+1}D(t) = O(t^{m+1}\ln t)$ since $D(t) = O(\ln t)$, however, in what follows we show that $H(t), L(t) = O(t^{m+3/2})$ and therefore we can omit this additional remainder as it will be adsorbed into this asymptotics.),

$$H(t) = \sum_{j=1}^{[\sqrt{t}]} F(t,j)D(t - j^2) + O(t^{m+1}\ln t), \tag{6.5.16}$$

$$L(t) = \sum_{j=1}^{[\sqrt{t}]} F(t,j)D((t - j^2)/4) + O(t^{m+1}\ln t), \tag{6.5.17}$$

and $F(t,j) = 2r(t,j)^{-2}\mathrm{Vol}(\mathbb{S}^3)^{-1} \int_{\mathbb{B}_{r(t,j)}} f(x,j)\,dx$. Moreover we have

$$H(t) = \sum_{j=1}^{[\sqrt{t}]-1} \left[F(t,j) - F(t,j+1)\right] M(t,j) + F(t,[\sqrt{t}])M([t,\sqrt{t}]) \tag{6.5.18}$$

with $M(t,n) = \sum_{i=1}^{n} D(t-i^2)$. This particular decomposition of $H(t)$ lends itself favourably to estimates for large t. Towards this end let us begin by noting $M(t,n) = O(\sqrt{t})$ (*see* [32] pp. 97). Furthermore a direct calculation gives

$$F(t,j) = \sum_{k=0}^{m} \binom{m}{k} (k+2)^{-1} j^{2(m-k)} (t-j^2)^{k+1}. \tag{6.5.19}$$

Thus $F(t,[\sqrt{t}])M(t,[\sqrt{t}]) = O(t^{m+3/2})$. Additionally a straightforward application of the binomial expansions in (6.5.19) leads to,

$$F(t,j) - F(t,j+1) = \sum_{k=1}^{m} \binom{m}{k} (k+2)^{-1} \sum_{i=0}^{k+1} \sum_{l=0}^{b_{i,k}} c_{i,k,l} j^l t^{k-i+1}, \tag{6.5.20}$$

where $b_{i,k} = 2i + 2(m-k) - 1$ and

$$c_{i,k,l} = (-1)^{i+1} \binom{k+1}{i} \binom{2i+2(m-k)}{k}.$$

With (6.5.20) at hand and again $M(t,n) = O(\sqrt{t})$ we obtain $H(t) = O(t^{m+3/2})$. Then in a similar fashion to the above $L(t) = O(t^{m+3/2})$ which in conjunction with (6.5.15) gives

$$\mathscr{E}_5(R) = \sum_{j=-R}^{R} \int_{\mathbb{B}_{r(R^2,j)}} f(x,j)\,dx + O(R^{2m+3}). \tag{6.5.21}$$

To complete the proof for the base case $n = 5$ we are left with showing that the leading term is as in (6.5.7). To achieve this we first apply the classical Euler-Maclaurin summation formula, namely,

$$\sum_{j=-R}^{R} g(j) = \int_{-R}^{R} g(y)\,dy + \int_{-R}^{R} \dot{g}(y)\psi(y)\,dy, \qquad g(y) = \int_{\mathbb{B}_{r(R^2,y)}} f(x,y)\,dx. \tag{6.5.22}$$

Now recall that $f(x,y) = (|x|^2 + y^2)^m$ and so from a straightforward calculation

$$g(y) = 2\pi^2 \left[\frac{R^{2m+4}}{m+2} + \frac{y^{2m+4}}{(m+1)(m+2)} - \frac{y^2 R^{2m+2}}{m+1} \right], \tag{6.5.23}$$

$$\dot{g}(y) = 4\pi^2 \left[\frac{y^{2m+3} - yR^{2m+2}}{m+1} \right]. \tag{6.5.24}$$

Hence applying the Mean-value theorem to the second integral in (6.5.22) gives

$$\begin{aligned}
\int_{-R}^{R} \dot{g}(y)\psi(y)\,dy &= \frac{4\pi^2}{m+1} \int_{-R}^{R} \left[y^{2m+3} - yR^{2m+2} \right] \psi(y)dy \\
&= \frac{4\pi^2 R^{2m+3}}{m+1} \left[\int_{\xi_1}^{R} \psi(y)dy - \int_{\xi_2}^{R} \psi(y)dy \right] \\
&= \frac{4\pi^2 R^{2m+3}}{m+1} \int_{\xi_1}^{\xi_2} \psi(y)dy = O(R^{2m+3}),
\end{aligned} \tag{6.5.25}$$

(here assuming without loss of generality that $-R < \xi_1 \leq \xi_2 < R$). Then setting $\omega = (x, y)$,

$$\int_{-R}^{R} g(y)\, dy = \int_{-R}^{R} \int_{\mathbb{B}^4_{r(R^2,y)}} (|x|^2 + y^2)^m \, dx\, dy = \int_{\mathbb{B}^5_R} |\omega|^{2m} \, d\omega = \frac{\mathrm{Vol}(\mathbb{S}^4)}{2m+5} R^{2m+5}.$$

Thus summarising we have succeeded in proving the base case

$$\mathscr{E}_5(R) = \frac{\mathrm{Vol}(\mathbb{S}^4)}{2m+5} R^{2m+5} + O(R^{2m+3}). \tag{6.5.26}$$

Hence we can prove (6.5.7) for $n \geq 5$ by induction. Indeed assume (6.5.7) is true for n. Then as before

$$\begin{aligned}\mathscr{E}_{n+1}(R) &= \sum_{j=-R}^{R} \sum_{w \in \mathbb{Z}^n} \left(|x|^2 + j^2\right)^m \chi_{\sqrt{R^2 - j^2}}(w) \\ &= \sum_{j=-R}^{R} \int_{\mathbb{B}^n_{r(R^2,j)}} f(w,j) dw + O\left(\sum_{j=1}^{R} (R^2 - j^2)^{2m+n-2}\right) \\ &= \frac{\mathrm{Vol}(\mathbb{S}^n)}{2m+n+1} R^{2m+n+1} + O\left(R^{2m+n-1}\right). \end{aligned} \tag{6.5.27}$$

Therefore (6.5.7) holds for $n+1$ and thus is true for all $n \geq 5$. Note that the leading term is obtained in exactly the same way as the base case $n = 5$ by applying the Euler-Maclaurin summation formula and the Mean value theorem. □

Therefore combining Theorem 6.7.1 with (6.5.6) gives that for $n \geq 5$,

$$\mathscr{N}(\lambda) = \kappa \frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{(2m+n)|W|} \left(\lambda + |\rho|^2\right)^{m+\frac{n}{2}} + O(\lambda^{m+\frac{n-2}{2}}) + O(\lambda^{m+\frac{n+2}{4}-\alpha}). \tag{6.5.28}$$

The proof of Theorem 6.5.1 is now a consequence of $\alpha = 1/4 - \varepsilon$ for any $\varepsilon > 0$ and therefore the sharp result holds when $n \geq 8$ since $(n+4)/2 - 2\alpha \leq n - 2$. This value for α comes from Proposition 11.4 on page 200 of [48]. Here we would like to point out that we have kept α undefined until the end of the proof to highlight how Theorem 6.5.1 can be improved, in terms of the rank n giving the sharp result, if the the value of α can be improved. Also we note that this can be done for $\mathbf{SO}(N)$ and $\mathbf{U}(N)$ since the lattice of analytic weights here is realised as $\mathbb{Z}^n$ whilst the multiplicity function is a homogeneous polynomial of even degree. (Let us also note here that in the cases of $\mathbf{SO}(2n+1)$ and $\mathbf{U}(2n)$ we need to define $\mathscr{E}$ to be the counting function over the shifted lattice $\mathbb{Z}^n + (1/2, \ldots, 1/2)$. In this case the counting function $\mathscr{E}$ has the same asymptotics as Theorem 6.7.1 with the proof working in the same fashion. This proof is completed in the appendix for the ease of the reader.) The case of $\mathbf{Spin}(N)$ results from $\mathbf{SO}(N)$ as in the proof of Theorem 6.4.3.

The leading coefficient can be written in the following form:

$$\begin{aligned}\kappa\frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{(2m+n)|W|} &= \frac{1}{(2m+n)|W|}\int_{\mathbb{S}^{n-1}} P(\theta)d\mathcal{H}^{n-1}(\theta) \\ &= \frac{1}{|W|}\int_0^1\int_{\mathbb{S}^{n-1}} P(\theta)d\mathcal{H}^{n-1}(\theta)r^{2m+n-1}dr \\ &= \frac{1}{|W|}\int_{\mathbb{B}_1^n} m_n(x)dx = \frac{\mathrm{Vol}\,(\mathbb{G})\,\omega_d}{(2\pi)^d}. \end{aligned} \tag{6.5.29}$$

Hence we have completed the proof of Theorem 6.5.1. □

6.6 Appendix 1: Asymptotics of weighted integrals involving Bessel functions

In this appendix we present the proof of an estimate used earlier in the paper. This is concerned with the asymptotics of Bessel functions and their weighted integrals.

Proposition 6.6.1. Let $\alpha \geq 2$ and $\beta > -1/2$. Then there exist constants $M > 0$ and $c > 0$ such that for any $z \geq M$ we have

$$\int_0^z t^{\alpha+\beta} J_\beta(t)\,dt \leq cz^{\alpha+\beta-\frac{1}{2}}. \tag{6.6.1}$$

Proof. Using the identity established in Lemma 6.6.1 below we can write

$$\int_0^z t^{\alpha+\beta} J_\beta(t)\,dt = z^{\alpha+\beta}J_{\beta+1}(z) - (\alpha-1)\int_0^z t^{\alpha+\beta-1}J_{\beta+1}(t)\,dt.$$

Next in virtue of the asymptotic decay of Bessel function at infinity (*see*, e.g., Stein & Weiss[72]) it follows that there exists $M > 0$ such that $J_{\beta+1}(z) \leq cz^{-\frac{1}{2}}$ for $z > M$. Hence we can write

$$\int_0^z t^{\alpha+\beta} J_\beta(t)\,dt \leq c_1 z^{\alpha+\beta-\frac{1}{2}} + c_2\int_M^z t^{\alpha+\beta-\frac{3}{2}}\,dt$$

from which the conclusion follows at once. □

Lemma 6.6.1. *Let $\alpha \geq 2$ and $\beta > -1/2$ with $z \in \mathbb{R}$. Then the following identity holds*

$$\int_0^z t^{\alpha+\beta} J_\beta(t)\,dt = z^{\alpha+\beta}J_{\beta+1}(z) - (\alpha-1)\int_0^z t^{\alpha+\beta-1}J_{\beta+1}(t)\,dt \tag{6.6.2}$$

Proof. Starting from the following weighted integral identity for Bessel functions (*cf.*, e.g., Grafakos [37] Appendix B.3)

$$\int_0^z J_{\beta-1}(t)t^\beta\,dt = z^\beta J_\beta(z)$$

we can write

$$\int_0^z t^{\alpha+\beta-1} J_{\beta+1}(t)dt = \int_0^z t^{\alpha-2} \int_0^t s^{\beta+1} J_\beta(s)\, dsdt = \int_0^z s^{\beta+1} J_\beta(s) \int_s^z t^{\alpha-2}\, dtds.$$

Now integrating the term on the right gives

$$\begin{aligned}\int_0^z t^{\alpha+\beta-1} J_{\beta+1}(t)\, dt &= \frac{1}{\alpha-1}\int_0^z s^{\beta+1} J_\beta(s)[z^{\alpha-1} - s^{\alpha-1}]\, ds \\ &= \frac{1}{\alpha-1} z^{\alpha+\beta} J_{\beta+1}(z) - \frac{1}{\alpha-1}\int_0^z s^{\beta+\alpha} J_\beta(s)\, ds\end{aligned}$$

and so the conclusion follows by simple manipulation of the above. □

6.7 Appendix 2: Case of the shifted lattice

Here we shall prove the asymptotics of the spectral counting function $\mathscr{N}(\lambda)$ when the underlying lattice is $\mathbb{Z}^n + 1/2$. Firstly, however we shall prove an analogue result of Theorem 6.7.1. Namely,

Theorem 6.7.1. Let $\mathscr{E}^s = \mathscr{E}^s(R)$ and $\mathscr{E}^o = \mathscr{E}^o(R)$ be the counting functions given by, (below $\mathbb{Z}^n_{odd}$ denotes the lattice of odd integers)

$$\mathscr{E}^s(R) = \sum_{x\in\mathbb{Z}^n+1/2} |x|^{2m}\chi_R(x), \text{ and } \mathscr{E}^o(R) = \sum_{x\in\mathbb{Z}^n_{odd}} |x|^{2m}\chi_R(x). \qquad (6.7.1)$$

Then provided $n \geq 5$ we have the asymptotics

$$\mathscr{E}^s(R) = \frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{2m+n} R^{2m+n} + O(R^{2m+n-2}), \qquad R \nearrow \infty, \qquad (6.7.2)$$

and,

$$\mathscr{E}^o(R) = \frac{1}{2^n}\frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{2m+n} R^{2m+n} + O(R^{2m+n-2}), \qquad R \nearrow \infty, \qquad (6.7.3)$$

Proof. To begin the proof we firstly observe,

$$\mathscr{E}^s(R) = \sum_{x\in\mathbb{Z}^n+1/2} |x|^{2m}\chi_R(x) = 2^{-2m}\sum_{x\in\mathbb{Z}^n_{odd}} |x|^{2m}\chi_{2R}(x) = 2^{-2m}\mathscr{E}^o(2R). \qquad (6.7.4)$$

Therefore by proving the asymptotics (6.7.3) for $\mathscr{E}^o$ we will, as a consequence of (6.7.2), have shown $\mathscr{E}^s$ has asymptotics of (6.7.2).

$$\mathscr{E}^o(R) = \sum_{x\in\mathbb{Z}^n_{odd}} |x|^{2m}\chi_R(x). \qquad (6.7.5)$$

Hence we can focus on $\mathscr{E}^o$ and in particular begin with the case $n = 4$ where the counting function $\mathscr{E}^o(R)$ can be written as,

$$\mathscr{E}^o_4(R) = \sum_{x\in\mathbb{Z}^n_{odd}} |x|^{2m}\chi_R(x) = \sum_{k=1}^{R^2} k^m r_4^*(k), \qquad (6.7.6)$$

where $r_4^*(k) = |\{x \in \mathbb{Z}^4_{odd} : |x|^2 = k\}|$. It is known that,

$$r_4^*(k) = \begin{cases} 16\sigma(l) & \text{if } k = 4l \text{ where } l \text{ odd,} \\ 0 & \text{otherwise,} \end{cases} \tag{6.7.7}$$

where above σ denotes the classical divisor function. Hence $\mathscr{E}_4^o$ becomes,

$$\begin{aligned} \mathscr{E}_4^o(R) &= 4^{m+2} \sum_{2l+1 \leq R^2/4} (2l+1)^m \sigma(2l+1) \\ &= 4^{m+2} \left[\mathscr{E}_4(R/2)/8 - 3 \times 2^m \sum_{k=1}^{R^2/8} k^m \sigma_{odd}(k) \right]. \end{aligned} \tag{6.7.8}$$

Above we have used the fact that counting function over the larger lattice $\mathbb{Z}^n$ can be written as,

$$\begin{aligned} \mathscr{E}_4(R) &= \sum_{x \in \mathbb{Z}^n} |x|^{2m} \chi_R(x) = \sum_{k=1}^{R^2} k^m r_4(k) \\ &= 8 \sum_{2k+1 \leq R^2} (2k+1)^m \sigma(2k+1) + 24 \sum_{2k \leq R^2} (2k)^m \sigma_{odd}(2k) \\ &= 8 \sum_{2k+1 \leq R^2} (2k+1)^m \sigma(2k+1) + 24 \times 2^m \sum_{k \leq R^2/2} k^m \sigma_{odd}(k). \end{aligned} \tag{6.7.9}$$

Then we recall from (6.5.8) in our previous work that,

$$\begin{aligned} \mathscr{E}_4^o(R) = &\int_{\mathbb{B}_R} |x|^{2m} \, dx + \\ &\frac{16}{R^2 \mathrm{Vol}(\mathbb{S}^3)} \int_{\mathbb{B}_R} |x|^{2m} dx \cdot \left[D(R^2) - D(R^2/4) \right] + O(R^{2m+2}), \end{aligned} \tag{6.7.10}$$

where $D(t) = \sum_{k \leq t} k^{-1} \psi(t/k)$ with $\psi(t/k) = t/k - [t/k] - 1/2$. Additionally we know from our previous work that,

$$\begin{aligned} \sum_{k=1}^{R^2/8} k^m \sigma_{odd}(k) &= S_m(R^2/8) - 2^{m+1} S_m(R^2/16) \\ &= \frac{R^{2m+4}}{8^{m+2}(m+2)} \left[\sum_{k=1}^{R^2/8} k^{-2} - \sum_{k=1}^{R^2/16} k^{-2} \right] \\ &\quad - \frac{R^{2m+2}}{8^{m+1}(m+2)} \left[D(R^2/8) - D(R^2/16) \right] + O(R^{2m+2}), \end{aligned} \tag{6.7.11}$$

Now using that $\sum_{k \leq t} k^{-2} = \pi^2/6 + O(t^{-1})$ we obtain that,

$$\begin{aligned} \sum_{k=1}^{R^2/8} k^m \sigma_{odd}(k) &= \frac{\pi^2 R^{2m+4}}{3 \times 2^{3m+8}(m+2)} \\ &\quad - \frac{R^{2m+2}}{8^{m+1}(m+2)} \left[D(R^2/8) - D(R^2/16) \right] + O(R^{2m+2}), \end{aligned} \tag{6.7.12}$$

At this point we note that,

$$\begin{aligned}\mathscr{E}_4^o(R/2)/8 =& \frac{1}{2^{2m+7}}F(R,0)+ \\ & \frac{1}{2^{2m+1}R^2\mathrm{Vol}(\mathbb{S}^3)}F(R,0)\cdot\left[D(R^2/4)-D(R^2/16)\right]+O(R^{2m+2}),\end{aligned} \tag{6.7.13}$$

where we have let

$$F(R,j)=\int_{\mathbb{B}_r(R^2,j)}(|x|^2+j^2)^m\,dx, \tag{6.7.14}$$

with $r(R^2,j)=\sqrt{R^2-j^2}$. We also note that

$$\frac{\pi^2 R^{2m+4}}{3\times 2^{3m+8}(m+2)}=\frac{1}{3\times 2^{m+4}}F(R,0). \tag{6.7.15}$$

Hence combining this with the above gives that,

$$\begin{aligned}\mathscr{E}_4^o(R)=&\frac{F(R,0)}{2^4}+\frac{F(R,0)}{R^2}\cdot\left[c_1D(R^2/4)+c_2D(R^2/8)+c_3D(R^2/16)\right] \\ &+O(R^{2m+2}),\end{aligned} \tag{6.7.16}$$

As before we now move onto the five dimensional case where $\mathscr{E}_5^o(R)$ can be written as,

$$\mathscr{E}_5^o(R)=\sum_{\substack{j=-R\\ odd}}^{R}\sum_{\mathbb{Z}^4_{odd}}(|x|^2+j^2)^m\chi_{r(R^2,j)}(x), \tag{6.7.17}$$

and since $(|x|^2+j^2)^m$ is just the linear combination of radial weights we obtain that,

$$\begin{aligned}\mathscr{E}_5^o(R)&=\sum_{\substack{j=-R\\ odd}}^{R}\mathscr{E}_4^o(r(R^2,j),F(R,j)) \\ &=\sum_{j=-R}^{R}\mathscr{E}_4^o(r(R^2,j),F(R,j))-\sum_{j=-R/2}^{R/2}\mathscr{E}_4^o(r(R^2,2j),F(R,2j)).\end{aligned} \tag{6.7.18}$$

Above we have defined,

$$\mathscr{E}_4^o(r(R^2,j),F(R,j))=\sum_{x\in\mathbb{Z}^4_{odd}}F(R^2,j)\chi_{r(R^2,j)}(x). \tag{6.7.19}$$

Note that $r(R^2,2j)=2r(R^2/4,j)$ and $F(R^2,2j)=2^{2n+4}F(R/2,j)$ so,

$$\mathscr{E}_5^o(R)=\sum_{j=-R}^{R}\mathscr{E}_4^o(r(R^2,j),F(R,j))-2^{2m+4}\sum_{j=-R/2}^{R/2}\mathscr{E}_4^o(2r(R^2/4,j),F(R/2,j)). \tag{6.7.20}$$

Hence we are left to find what is the asymptotics of,

$$\mathscr{F}_c(t)=\sum_{j=-t}^{t}\mathscr{E}_4^o(cr(t^2,j),F(t,j)) \tag{6.7.21}$$

as $t \to \infty$ where $c = 1, 2$. To this end we note that,

$$\mathscr{F}_c(t) = \sum_{j=-t}^{t} \frac{F(t,j)}{2^4} + c_1 H(t, 4/c) + c_2 H(t, 8/c) + c_3 H(t, 16/c) + O(t^{2m+3}) \tag{6.7.22}$$

where we have defined $H(t, \xi)$ in the following way,

$$H(t,\xi) = \sum_{j=-t}^{t} \frac{F(t,j)}{r(t^2,j)} D\left(\frac{t^2 - j^2}{\xi}\right) \tag{6.7.23}$$

It follows easily from our previous work (6.5.16)-(6.5.21) that $H(t, \xi) = O(t^{2m+3})$ and therefore,

$$\mathscr{F}_c(t) = \sum_{j=-t}^{t} \frac{F(t,j)}{2^4} + O(t^{2m+3}) \tag{6.7.24}$$

Then also as in (6.5.22)-(6.5.26) we know that the leading term in the above is given by,

$$\frac{1}{2^4} \frac{\mathrm{Vol}(\mathbb{S}^4)}{2m+5} t^{2m+5} + O(t^{2m+3}) \tag{6.7.25}$$

and therefore we obtain that,

$$\mathscr{F}_c(t) = \frac{1}{2^4} \frac{\mathrm{Vol}(\mathbb{S}^4)}{2m+5} t^{2m+5} + O(t^{2m+3}) \tag{6.7.26}$$

as $t \to \infty$ for $c = 1$ or $c = 2$. Thus,

$$\mathscr{E}_5^o(R) = \mathscr{F}_1(R) - 2^{2m+4}\mathscr{F}_2(R/2) = \frac{1}{2^5} \frac{\mathrm{Vol}(\mathbb{S}^4)}{2m+5} R^{2m+5} + O(R^{2m+3}), \tag{6.7.27}$$

Hence we have shown the result for $n = 5$ and as before $n > 5$ follows by induction. Indeed assume (6.7.3) is true for some fixed $n \geq 5$. Then as before

$$\begin{aligned}
\mathscr{E}_{n+1}^o(R) &= \sum_{j=-R}^{R} \mathscr{E}_n^o(r(R^2,j), F(R,j)) - \sum_{j=-R/2}^{R/2} \mathscr{E}_n^o(r(R^2,2j), F(R,2j)) \\
&= \sum_{j=-R}^{R} \frac{F(R,j)}{2^n} - 2^{2m+n} \sum_{j=-R/2}^{R/2} \frac{F(R/2,j)}{2^n} + O\left(\sum_{j=1}^{R} (R^2 - j^2)^{2m+n-2}\right) \\
&= \frac{1}{2^{n+1}} \frac{\mathrm{Vol}(\mathbb{S}^n)}{2m+n+1} R^{2m+n+1} + O\left(R^{2m+n-1}\right).
\end{aligned} \tag{6.7.28}$$

Therefore (6.7.3) holds for $n+1$ and thus is true for all $n \geq 5$ which because of the identity (6.7.4) completes our proof. $\square$

Now we recall that in this situtation of the shifted lattice $\mathbb{Z}^n + (1/2, \ldots, 1/2)$ we can write the spectral counting function $\mathscr{N}(\lambda)$ as,

$$\begin{aligned}
\mathscr{N}(\lambda) &= \frac{1}{|W|} \sum_{x \in \mathbb{Z}^n + 1/2} m_n(x) \chi_R(x) = \frac{1}{2^{2m}|W|} \sum_{x \in \mathbb{Z}^n_{odd}} m_n(x) \chi_{2R}(x) \\
&= \frac{1}{2^{2m}|W|} \sum_{k=1}^{4R^2} k^m \sum_{\theta_j \in H_k^*} P(\theta_j)
\end{aligned} \tag{6.7.29}$$

where $H_k^* = \{\theta = x/|x| : x \in \mathbb{Z}^n_{odd} \text{ and } |x|^2 = k\}$. Now proceeding as before we know that,

$$|\mathscr{N}(\lambda) - (2^{2m}|W|)^{-1}\kappa\,\mathscr{E}^o(2R)| \leq \frac{1}{2^{2m}|W|}\sum_{k=1}^{4R^2} k^m \left| \sum_{\theta_j \in H_k^*} P(\theta_j) - \kappa r_n^*(k) \right| \tag{6.7.30}$$

Furthermore it is possible to use Proposition 11.4 on page 200 of [48] to prove that,

$$\left| \sum_{\theta_j \in H_k^*} P(\theta_j) - \kappa r_n^*(k) \right| = O(k^{\frac{n}{4}-\alpha}), \tag{6.7.31}$$

with $\alpha = 1/4 - \varepsilon$ for any $\varepsilon > 0$ as before. Thus, like in the previous case we conclude that, (recall that $R = \sqrt{\lambda + |\rho|^2}$),

$$\mathscr{N}(\lambda; \mathbb{G}) = \frac{\omega_d \mathrm{Vol}(\mathbb{G})}{(2\pi)^d}\lambda^{\frac{d}{2}} + O(\lambda^{\frac{d-2}{2}+\beta}), \qquad \lambda \nearrow \infty. \tag{6.7.32}$$

where β is given by

$$\beta = \begin{cases} 2 - \frac{n}{4} - \alpha & \text{if } n = 5, 6, 7, \\ 0 & \text{if } n \geq 8 \end{cases}.$$

and $\alpha = 1/4 - \varepsilon$ for any $\varepsilon > 0$.

Chapter 7

Discussion and final remarks

The work in this thesis has made modest progress in two seemly distinct, but connected, areas of mathematics. Namely through the analysis in Chapters 2 - 5 we have lead the way to better understanding the symmetry properties of certain L^1-local minimisers. This in particular improves upon the previous work in [64, 65, 66, 74, 75, 76] and it is our hope that some of the symmetrisation techniques which were developed to tackle this problem will find a place in other settings as well. Through this study we have also made in the planar case small contribution to the understanding of *global invertibility* for mappings in $\mathcal{A}(\mathbf{X})$ as well as the *regularity* of minimisers in homotopy classes $\mathscr{A}_k$ for a wide variety of energy functionals.

Furthermore in Chapter 6 we tackled the problem of studying the asymptotic behaviour of the spectral counting function for the compact Lie groups $\mathbf{SO}(n), \mathbf{SU}(n), \mathbf{Spin}(n)$ and $\mathbf{U}(n)$. Here the work improves on the previously known results for these groups and accumulates in presenting the sharp asymptotics $\mathbf{SO}(n), \mathbf{Spin}(n)$ and $\mathbf{U}(n)$ when the rank of the corresponding group is strictly larger than seven. These results provide further evidence that the lack of periodic geodesics a Riemannian manifold (M^d, g) possess substantially effects the sharpness in the reminder term of the famous Avakumovic-Hörmander result. Furthermore, it is our hope that these results will motivate further study into the asymptotics of a general compact Lie group and whether results in a similar vein carry over into this setting.

7.0.1 Future research

As alluded to above there is still much progress to be made on both fronts of this thesis. One natural line of research to continue from Chapters 2 - 5 is to try and extend the symmetrisation methods into higher dimensions, namely when $n > 2$. It would be interesting to see if anything of this nature is possible and whether in particular any more light can be shed on the *odd* dimensional case. Moreover, it would be interesting if the assumptions required on the energy functional from Chapter 4 and 5 to provide *twist* solutions in homotopy classes can be weakened.

With regard to the results from Chapter 6 the obvious path of future research is to see if the obtained result can be extended in the following two ways: Firstly is it possible to obtain for the groups mentioned above analogue results to the classical lattice point counting problem when the rank of the groups are between 2 and 4. By this we mean are similar remainder terms available for these Lie groups spectral counting function as there is for the classical Gauss circle problem and its higher dimensional analogues. Further to this one can also ask do the results obtained extend to all compact Lie groups and moreover to all symmetric spaces whose rank bigger is strictly bigger than one.

Bibliography

[1] S.S. Antman, *Nonlinear problems of Elasticity*, Appl. Math. Sci., **107**, 2nd Edition, Springer, 2005.

[2] K. Astala, T. Iwaniec, G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Math. Ser., **48**, Princeton University Press, 2009.

[3] K. Astala, T. Iwaniec, G. Martin, J. Onninen, Extremal mappings of finite distortion, *Proc. Lond. Math. Soc.*, **91**, 2005, pp. 655-702.

[4] V.G. Avakumovic, Uber die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten, *Math. Z.*, **65**, 1956, pp. 327-344.

[5] R.O. Awonusika, A. Taheri, *Spectral Invariants on Compact Symmetric Spaces: From Heat Trace to Functional Determinants*, Submitted 2016, (pp. 262+ix).

[6] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Ration. Mech. Anal.*, **63**, 1977, pp. 337-403.

[7] J.M. Ball, Constitutive inequalities and existence theorems in nonlinear elastostatics, *Nonlinear analysis and mechanics: Heriot-Watt Symposium*, **1**, (Ed. R.J. Knops), Pitman, London, 1977.

[8] J.M. Ball, Global invertibility of Sobolev functions and the interpenetration of matter, *Proc. Roy. Soc. Edin. A*, **88**, 1981, pp. 315-328.

[9] J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Phil. Trans. Roy. Soc. Lond. A*, **306**, 1982, pp. 557-611.

[10] J.M. Ball, Differentiability properties of symmetric and isotropic functions, *Duke Math. J.*, **51**, No. **3**, 1984, pp. 699-728.

[11] J.M. Ball, Some open problems in elasticity, *Geometry, mechanics and dynamics*, Springer, New York, 2002, pp. 3-59.

[12] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg-Landau Vortices*, Progress in Nonlinear Differential Equations and their Applications, **13**, Birkhäuser Boston, Inc., Boston, MA, 1994.

[13] F. Bethuel, X. Zheng, Density of smooth functions between two manifolds in Sobolev spaces. *J. Funct. Anal.*, **80**, 1988, pp. 60-75.

[14] P. Berard, On the wave equation on a compact Riemannian manifold without conjugate points, *Math. Z.*, **155**, 1977, pp. 249-276.

[15] J. Bourgain, H. Brezis, P. Mironescu, Lifting in Sobolev spaces, *J. Anal. Math.*, **80**, 2000, pp. 37-86.

[16] T. Bröckner, T. tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, **98**, Springer-Verlag, 1985.

[17] J.E. Brothers, W.P. Ziemer, Minimal rearrangements of Sobolev functions, *Acta Univ. Carolin. Math. Phys.*, **28**, 1987, pp 13-24.

[18] D. Bump, *Lie Groups*, Graduate Texts in Mathematics, Second Edition, **225**, Springer, 2013.

[19] L. Cesari, *Optimization - theory and application*, Applications of Mathematics, **17**, Springer, 1983.

[20] F. Chamizo, H. Iwaniec, On the sphere problem, *Rev. Mat. Iberoamericana*, **11**, 1995, pp. 417-429.

[21] J. Chazarain, Formule de Poisson pour les variétés riemanniennes, *Invent. Math.*, **24**, 1974, pp. 65-82.

[22] A. Cianchi, N. Fusco, Functions of bounded variation and rearrangements, *Arch. Ration. Mech. Anal.*, **165**, 2002, pp. 1-40.

[23] P. Ciarlet, *Mathematical Elasticity: Three dimensional elasticity*, **1**, Stud. Math. Appl., North-Holland Publishing Co., Amsterdam, 1988.

[24] B. Dacorogna, *Direct methods in the Calculus of Variations*, Second Edition, Appl. Math. Sci., **78**, Springer, New York, 2008.

[25] G. Dal Maso, *Introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Applications, **8**, Bikhäuser, 1992.

[26] E. De Giorgi, Convergence problems for functionals and operators, in: *Proc. Int. meeting on recent methods in nonlinear analysis*, (Eds. E. De Giorgi *et. al.*), 1979, pp. 223-244.

[27] J.J. Duistermaat, L. Hörmander, Fourier Integral Operators **II**, *Acta Math.*, **128**, 1972, pp. 183-269.

[28] J.J. Duistermaat, V.W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.*, **29**, 1975, pp. 39-79.

[29] L.C. Evans, R.F. Gariepy, On the partial regularity of energy minimizing area preserving maps, *Calc. Var. & PDEs*, **63**, 1999, pp. 357-372.

[30] H. Federer, *Geometric measure theory*, Classics in Mathematics, **153**, Springer-Verlag, 1969.

[31] W. Freeden, *Metaharmonic lattice point theory*, Pure and Applied Mathematics, **299**, CRC Press, 2011.

[32] F. Fricker, *Einführung in die Gitterpunktlehre.*, Mathematische Reihe, **73**, Birkhäuser Verlag, Basel-Boston, Mass., 1982.

[33] I. Fonseca, W Gangbo, Local invertibility of Sobolev functions, *SIAM J. Math. Anal.*, **26**, 1995, pp. 280-304.

[34] I. Fonseca, W. Gangbo, *Degree theory in analysis and applications*, Oxford Lecture Ser. Math. Appl., Oxford University Press, 1995.

[35] V.M. Gol'dshtein, Yu.G. Reshetnyak, *Quasiconformal mappings and Sobolev spaces*, Math. Appl. (Soviet Ser.), Kluwer Academic Publishers, 1990.

[36] V.M. Gol'dshtein, S.K. Vodopyanov. Quasiconformal mappings and spaces of functions with generalized first derivatives, *Siberian Math. J.*, **17**, 1977, pp. 515-531.

[37] L. Grafakos, *Classical and modern Fourier analysis*, Graduate Texts in Mathematics, **249** & **250**, Springer, 2008.

[38] D.R. Heath-Brown, Lattice points in the sphere, *Number theory in progress*, **2**, de Gruyter, Berlin, 1999, pp. 883-892.

[39] J. Heinonen, P. Koskela, Sobolev mappings with integrable dilatations, *Arch. Ration. Mech. Anal.*, **125**, 1993, pp. 81-97.

[40] S. Helgason, *Groups and geometric analysis: Integral geometry, invariant differential operators and spherical functions*, Mathematical Surveys and Monographs, **83**, American Mathematical Soc., 1984.

[41] S. Hencl, C. Mora-Corral, Diffeomorphic approximation of continuous almost everywhere injective Sobolev deformations in the plane, *Q. J. Math.*, **66**, 2015, pp. 1055-1062.

[42] E. Hlawka, Über Integrale auf konvexen Körpern **I**, *Montash. Math.*, **54**, 1950, pp. 1-36.

[43] L. Hörmander, The spectral function of an elliptic operator, *Acta Math.*, **121**, 1968, pp. 193-218.

[44] L. Hörmander, Fourier integral operators **I**, *Acta Math.*, **127**, 1971, pp. 79-183.

[45] L. Hörmander, *The analysis of linear partial differential operators*, **1-4**, Classics in Mathematics, Springer, 2003.

[46] M.N. Huxley, Exponential sums and lattice points. III, *Proc. Lond. Math. Soc. (3)*, **87**, 2003, pp. 591-609.

[47] V. Ivrii, *Microlocal analysis and precise spectral asymptotics*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[48] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, **17**. American Mathematical Soc., 1997.

[49] T. Iwaniec, J. Onninen, n-harmonic mappings between annuli: the art of integrating free Lagrangians, *Mem. Amer. Math. Soc.*, **218**, 2012, (viii+105 pp.).

[50] T. Iwaniec, V. Sverak, On mappings with integrable dilatations, *Proc. Am. Math. Soc.*, **118**, 1993, pp. 181-188.

[51] A.W. Knapp, *Representation theory of semisimple groups: An overview based on examples.*, Princeton Math. Ser., **36**, Princeton University Press, 1986.

[52] B.M. Levitan, On the asymptotic behaviour of the spectral function of a self-adjoint differential equation of second order, *Isv. Akad. Nauk. SSSR Ser. Mat.*, **16**, 1952, pp. 325-352.

[53] J. Malý, D. Swanson, W. Ziemer, The co-area formula for Sobolev mappings, *Trans. Amer. Math. Soc.*, **355**, 2003, pp. 477-492.

[54] A.V. Malysev, On representations of integers by positive quadratic forms, *Trudy Mat. Inst. Steklov.*, **65**, 1962, pp. 1-212.

[55] H.P. McKean, I. Singer, Curvature and the eigenvalues of the Laplacian, *J. Diff. Geom.*, **1**, 1967, pp. 43-69.

[56] G. Meisters, C. Olech, Locally one-to-one mappings and a classical theorem on Schlicht functions. *Duke Math. J.*, **30**, 1963, pp. 63-80.

[57] C.B. Morrey. *Multiple integrals in the calculus of variations*, Classics in Mathematics, **130**, Springer, 1966.

[58] C. Morris, A. Taheri, Twists as energy minimisers in homotopy classes: symmetrisation and the co-area formula, Accepted 2017.

[59] C. Morris, A. Taheri, On the periodic orbits of the geodesic flow on a compact Lie group, the lattice of weights and the Cartan subalgebra, *In preparation*, 2016.

[60] C. Morris, A. Taheri, On the uniqueness of energy minimisers in homotopy classes, Submitted for publications, 2017.

[61] S. Müller, S.J. Spector, Q. Tang, Invertibility and a topological property of Sobolev maps, *SIAM J. Math. Anal.*, **27**, 1996, pp. 959-976.

[62] C. Pommerenke, Über die Gleichverteilung von Gitterpunkten auf m-dimensionalen Ellipsoiden, *Acta Arith.*, **5**, 1959, pp. 227-257.

[63] K. Post, J. Sivaloganathan, On homotopy conditions and the existence of multiple equilibria in finite elasticity, *Proc. Roy. Soc. Edin. A*, **127**, 1997, pp. 595-614.

[64] M.S. Shahrokhi-Dehkordi, A.Taheri, Generalised twists, stationary loops and the Dirichlet energy over a space of measure preserving maps, *Calc. Var. & PDEs*, **35**, 2009, pp. 191-213.

[65] M.S. Shahrokhi-Dehkordi, A.Taheri, Generalised twists, $\mathbf{SO}(n)$ and the p-energy over a space of measure preserving maps, *Ann. Inst. Henri Poincarè, Analyse non lineaire*, **26**, 2009, pp. 1897-1924.

[66] M.S. Shahrokhi-Dehkordi, A.Taheri, Polyconvexity, generalised twists and energy minimisers on the space of self-maps of annuli in the multi-dimensional calculus of variations, *Adv. Calc. Var.*, 2009, pp. 361-396.

[67] M.A. Shubin, *Pseudodifferential operators and spectral theory*, Second Edition, Springer-Verlag, Berlin, 2001.

[68] J. Sivaloganathan, S.J. Spector. On the symmetry of energy-minimising deformations in nonlinear elasticity, I. Incompressible materials, *Arch. Ration. Mech. Anal.*, **196**, 2010, pp. 363-394.

[69] J. Sivaloganathan, S.J. Spector. On the symmetry of energy-minimising deformations in nonlinear elasticity, II. Compressible materials, *Arch. Ration. Mech. Anal.*, **196**, 2010, pp. 395-431.

[70] C. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, **105**, Cambridge University Press, 1993.

[71] C. Sogge, *Hangzhou lectures on eigenfunctions of the Laplacian*, Annals of Mathematics Studies, Vol. **230**, Princeton University Press, 2014.

[72] E.M. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, 1971.

[73] V. Sverak, Regularity properties of deformations with finite energy, *Arch. Ration. Mech. Anal.*, **100**, 1988, pp. 105-127.

[74] A. Taheri, Local minimizers and quasiconvexity - the impact of Topology, *Arch. Ration. Mech. Anal.*, **176**, 2005, pp. 363-414.

[75] A. Taheri, Minimizing the Dirichlet energy over a space of measure preserving maps, *Top. Meth. Nonlinear Anal.*, **33**, 2009, pp. 179-204.

[76] A. Taheri, Homotopy classes of self-maps of annuli, generalised twists and spin degree, *Arch. Ration. Mech. Anal.*, **197**, 2010, pp. 239-270.

[77] A. Taheri, Spherical twists, stationary loops and harmonic maps from generalised annuli into spheres, *NoDEA*, **19**, 2012, pp. 79-95.

[78] A. Taheri, *Function spaces and partial differential equations*, **I** & **II**, Oxford Lecture Series in Mathematics and Its Applications, **40** & **41**, OUP, 2015.

[79] N.J. Vilenkin, *Special functions and the theory of group representations*, Translations of Mathematical Monographs, **22**, AMS, 1968.

[80] A. Walfisz, Über Gitterpunkte in vierdimensionalen Ellipsoiden, *Math. Z.*, **72**, 1959/1960, pp. 259-278.

[81] H. Weyl, Über die asymptotische Verteilung der Eigenwerte, *Nachrichten der Königlichen Gesellschaft der Wissenschaftenzu Göttingen, Math. physikal. Klasse*, 1911, pp. 110-117.

[82] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrauhlung), *Math. Ann.*, **71**, 1912, pp. 441-479.

[83] H. Weyl, Über die Abhängigkeit der Eigenschwingungen einer Membran und deren Begrenzung, *J. Reine Angew. Math.*, **141**, 1912, pp. 1-11.

[84] H. Weyl, *The classical groups: Their invariants and representations*, Princeton Landmarks in Mathematics, Princeton University Press, 1939.